Reading: Dummit–Foote Ch. 10.3, 10.5, & pp 911–913.

Summary of definitions and main results

Definitions we've covered: Linear independence, basis, free module, rank of a free module, universal property, category, object, morphism, functor, coproduct, abelianization, monomorphism, epimorphism, isomorphism, covariant and contravariant functors, forgetful functor, free functor, dual space functor, functors $\mathrm{Hom}_R(D, -)$ and $\mathrm{Hom}_R(-, D)$, exact, exact sequence, short exact sequence, extension of C by A, presentation.

Main results: universal property for free modules, construction of the free module $F(A)$, verification that $F(A)$ satisfies the universal property, universal properties define objects up to unique isomorphism, in the category R-mod monomorphisms are precisely the injections, free functor $F : \underline{\mathrm{Set}} \to R\text{–}\underline{\mathrm{Mod}}$ is a covariant functor, $\mathrm{Hom}_R(D, -)$ is a covariant functor.

Warm-Up Questions

The "warm-up" questions do not need to be submitted (and won't be graded).

1. Let $N_1, \ldots, N_k$ be R–modules, and let $M = N_1 \times N_2 \times \cdots \times N_k$ be their external direct sum. Show that M naturally contains isomorphic copies of each R–module N_i via the inclusion
$$N_i \xrightarrow{\cong} \Big(0 \times \cdots \times 0 \times N_i \times 0 \times \cdots \times 0\Big) \hookrightarrow M.$$
Show that M is the internal direct sum of these k submodules.

2. In this question, we will verify how the universal property defining free modules will fail for modules that are not free. Let $R = \mathbb{Z}$.
   (a) Consider the abelian group $\mathbb{Z}/m\mathbb{Z}$ and the subset $A = \{1 \pmod m\}$. Show that $\mathbb{Z}/m\mathbb{Z}$ fails to satisfy the universal property for being the free module on the basis A.
   Hint: Consider $M = \mathbb{Z}$ and any nonzero set map $A \to M$.
   (b) Consider the abelian group $\mathbb{Z}$ and the subset $A = \{2\}$. Show that $\mathbb{Z}$ fails to satisfy the universal property for being the free module on the basis A.
   Hint: Consider $M = \mathbb{Z}$ and the set map taking $2 \in A$ to $1 \in M$.
   (c) Consider the abelian group $\mathbb{Z} \oplus \mathbb{Z}$ and the subset $A = \{(1, 0)\}$. Show that $\mathbb{Z} \oplus \mathbb{Z}$ fails to satisfy the universal property for being the free module on the basis A.

3. (a) Let A be any finite set of n elements. Show that the free R–module on A is isomorphic as an R–module to R^n.
   (b) For R commutative, are the polynomial rings $R[x]$ and $R[x, y]$ free R–modules? What about Laurent polynomials $R[x, x^{-1}]$? Rational functions in x?
   (c) Do these arguments work for series $R[[x]]$?

4. (a) Show that $M = \mathbb{Z}/10\mathbb{Z} \oplus \mathbb{Z}/10\mathbb{Z}$ is a free $\mathbb{Z}/10\mathbb{Z}$–module by finding a basis.
   (b) Show that the element $(2, 2)$ cannot be an element of any basis for M.
   (c) Is the submodule $N = \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/10\mathbb{Z}$ also free?

5. Consider the free $\mathbb{Z}$–module $F = \mathbb{Z}^2$, and its submodule $N = 2F$. Is N a free $\mathbb{Z}$–module?

6. Show that $M = \mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$ is a rank-2 free module over $\mathbb{Z}/6\mathbb{Z}$, and find all possible pairs of elements $\{a, b\} \subset M$ that form a basis for M.

7. (a) Show that A is a basis for the R–module RA it generates if and only if A is R–linearly independent.
   (b) Find a counterexample to the following false statement: If M is a free R–module and $A \subseteq M$ is an R–linearly independent subset of M, then A can be extended to a basis for M.

8. (a) Let F be the free R–module on a set A. Show that if R has no zero divisors and $N \subseteq F$ is any nonzero submodule, then $\text{ann}(N) = \{0\}$.
   (b) Let $R = \mathbb{Z}/10\mathbb{Z}$ and let $F = R^2$ be the free R–module of rank 2. Compute the annihilator of the submodule $2F$.

9. In class (and in Dummit-Foote 10.3 Theorem 6) we gave a construction of a free module $F(A)$ on a set A. Verify that this construction is in fact a free module with basis A (as given in the definition on p354). Show moreover that $F(A) \cong \bigoplus_A R$.

10. (a) Citing results from linear algebra, explain why every vector space over a field $\mathbb{F}$ is a free $\mathbb{F}$–module.
    (b) When $\mathbb{F}$ is a field, any minimal finite generating set $B = \{a_1, \ldots, a_n\}$ of an $\mathbb{F}$–module must be linearly independent and therefore a basis. Prove that in general, if an R–module has a minimal generating set $B = \{a_1, \ldots, a_n\}$, then R need not be free on B.
    (c) Suppose that M is an R–module containing elements $\{a_1, a_2, \ldots, a_n\}$ such that $M = Ra_1 \oplus Ra_2 \oplus \cdots \oplus Ra_n$. Explain how $A = \{a_1, a_2, \ldots a_n\}$ could fail to be a basis for M. What conditions on the elements a_i could ensure that A is a basis?

11. (a) Prove that in the category of R–modules, a morphism is epic if and only if it is a surjective map.
    (b) Prove that in the category of rings, the map $\mathbb{Z} \to \mathbb{Q}$ is an epic morphism that is not surjective.

12. Let $\mathscr{C}$ be a category containing objects A and B, and let F be a functor $F : \mathscr{C} \to \mathscr{D}$. Show that if A and B are isomorphic objects of $\mathscr{C}$, then $F(A)$ and $F(B)$ will be isomorphic objects of $\mathscr{D}$.

13. Given a group G, define a category $\mathscr{G}$ with a single object $\bigstar$ and morphisms $\text{Hom}_{\mathscr{G}}(\bigstar, \bigstar) = \{g \mid g \in G\}$. The composition law is given by the group operation. Show that a function between groups $G \to H$ is a group homomorphism if and only if the corresponding map between categories $\mathscr{G} \to \mathscr{H}$ is a functor.

14. Let $\underline{\text{f Set}}$ denote the category of finite sets and all functions between sets. Let $\mathscr{P} : \underline{\text{f Set}} \to \underline{\text{f Set}}$ be the function that takes a finite set A to its *power set* $\mathscr{P}(A)$, the set of all subsets of A. If $f : A \to B$ is a function of finite sets, let $\mathscr{P}(f) : \mathscr{P}(A) \to \mathscr{P}(B)$ be the function that takes a subset $U \subseteq A$ to the subset $f(U) \subseteq B$.
    (a) Show that $\mathscr{P}$ is a covariant functor.
    (b) What if we had instead defined $\mathscr{P}(f) : \mathscr{P}(B) \to \mathscr{P}(A)$ to take a subset $U \subseteq B$ to its preimage $f^{-1}(U) \subseteq A$ under f?

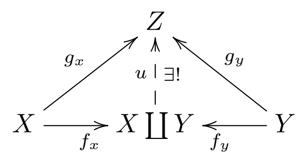
15. Let 0 denote the trivial abelian group. Give an example of a functor $F : \underline{\text{Ab}} \to \underline{\text{Ab}}$ such that $F(0) = 0$, and a functor $F : \underline{\text{Ab}} \to \underline{\text{Ab}}$ such that $F(0) \neq 0$.

16. Write down short exact sequences giving presentations of the following R–modules M. Give a list of generators and relations for M.
    (a) R^n
    (b) $R = \mathbb{Z}$, $M = \mathbb{Z}/10\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$
    (c) $R = \mathbb{Q}$, $M = \mathbb{Q}[x]/\langle x^2 + 1 \rangle$
    (d) $R = \mathbb{C}[x, y]$, $M = \langle x, y \rangle$

17. (a) Find two non-isomorphic extensions of $\mathbb{Z}$–modules $\mathbb{Z}/n\mathbb{Z}$ by $\mathbb{Z}$.
    (b) Find two non-isomorphic extensions of $\mathbb{Z}$–modules $\mathbb{Z}/n\mathbb{Z}$ by $\mathbb{Z}/n\mathbb{Z}$.
    (c) How many extensions of $\mathbb{Z}$ by $\mathbb{Z}/n\mathbb{Z}$ can you find?
    (d) Show that if $0 \to U \to W \to V \to 0$ is a short exact sequence of vector spaces, then $W \cong V \oplus U$.

Assignment Questions

1. **(Coproducts).** Let $\mathcal{C}$ be a category with objects X and Y. The *coproduct* of X and Y (if it exists) is an object $X \coprod Y$ in $\mathcal{C}$ with maps $f_x : X \to X \coprod Y$ and $f_y : Y \to X \coprod Y$ satisfying the following universal property: whenever there is an object Z with maps $g_x : X \to Z$ and $g_y : Y \to Z$, there exists a unique map $u : X \coprod Y \to Z$ that makes the following diagram commute:



   (a) Let X and Y be objects in $\mathcal{C}$. Show that, if the coproduct $(X \coprod Y, f_x, f_y)$ exists in $\mathcal{C}$, then the universal property determines it uniquely up to unique isomorphism.

   (b) Prove that in the category of R–modules, the coproduct of R–modules $X \coprod Y$ is $X \oplus Y$ with the canonical inclusions of X and Y. In other words, this universal property defines the direct sum operation on R–modules.

   (c) Explain how to reinterpret this universal property for the direct sum of R–modules as a bijection of sets
$$\mathrm{Hom}_R(X \oplus Y, Z) \cong \mathrm{Hom}_R(X, Z) \times \mathrm{Hom}_R(Y, Z)$$
   for R–modules X, Y, Z.

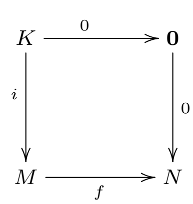
   (d) Prove that in the category of groups, the univeral property for the coproduct $X \coprod Y$ of groups X and Y does *not* define the direct product of those groups along with their canonical inclusions. (It is a construction called the *free product* of groups).

   (e) Prove that in the category of sets, the coproduct $X \coprod Y$ of sets X and Y is their disjoint union.

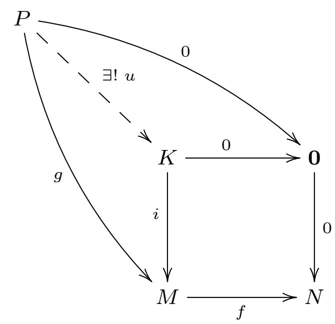
2. (a) A *zero object* $\mathbf{0}$ in a category is an object with the following property: For any object M, there is a unique morphism from M to $\mathbf{0}$, and a unique morphism from $\mathbf{0}$ to M. Show that if a category has a zero object, then it is unique up to unique isomorphism.

   (b) Let $\mathcal{C}$ be the category of R–modules, and show that the zero module $\{0\}$ is a zero object. This definition allows us to define the *zero map* 0 between R–modules M and N: it is the composition of the unique map $M \to \mathbf{0}$ with the unique map $\mathbf{0} \to N$.

   (c) Let $\mathcal{C}$ be the category of R–modules. If $f : M \to N$ is a morphism in $\mathcal{C}$, then define the *kernel* $i : K \to M$ of f to be the map i such that $f \circ i$ is the zero morphism 0



   and satisfying the following: whenever there is a map of R–modules $g : P \to M$ such that $f \circ g = 0$, there is a unique map $u : P \to K$ such that $i \circ u = g$. In other words, there is a unique map u that makes the following diagram commute.



Verify that the kernel of an R–module map (in the way we had previously defined it, as the preimage of 0) does indeed satisfy this universal property.

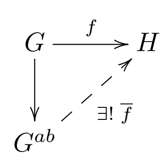
(d) Show that this univeral property determines the map $i : K \to M$ uniquely up to unique isomorphism. Conclude that we therefore can indeed take this universal property as the *definition* of the kernel of f.

3. **(Abelianization).** Let Grp denote the category of groups and group homomorphisms, and let Ab denote the category of abelian groups and group homomorphisms. Define the *abelianization* G^{ab} of a group G to be the quotient of G by its *commutator subgroup* $[G, G]$, the subgroup normally generated by *commutators*, elements of the form $ghg^{-1}h^{-1}$ for all $g, h \in G$.

   (a) Define a map of categories $[-, -] : \underline{\text{Grp}} \to \underline{\text{Grp}}$ that takes a group G to its commutator subgroup $[G, G]$, and a group morphism $f : G \to H$ to its restriction to $[G, G]$. Check that this map is well defined (ie, check that $f([G, G]) \subseteq [H, H]$) and verify that $[-, -]$ is a functor.

   (b) Show that G^{ab} is an abelian group. Show moreover that if G is abelian, then $G = G^{ab}$.

   (c) Show that the quotient map $G \to G^{ab}$ satisfies the following universal property: Given any **abelian** group H and group homomorphism $f : G \to H$, there is a unique group homomorphism $\overline{f} : G^{ab} \to H$ that makes the following diagram commute:



   This universal property shows that G^{ab} is in a sense the "largest" abelian quotient of G.

   (d) Show that the map ab that takes a group G to its abelianization G^{ab} can be made into a functor $ab : \underline{\text{Grp}} \to \underline{\text{Ab}}$ by explaining where it maps morphisms of groups $f : G \to H$, and verifying that it is functorial.

   (e) The category Ab is a subcategory of Grp. Define the functor $\mathcal{A} : \underline{\text{Ab}} \to \underline{\text{Grp}}$ to be the inclusion of this subcategory; $\mathcal{A}$ takes abelian groups and group homomorphisms in Ab to the same abelian groups and the same group homomorphisms in Grp. Briefly explain why the universal property in Part (c) can be rephrased as follows: Given groups $G \in \underline{\text{Grp}}$ and $H \in \underline{\text{Ab}}$, there is a natural bijection between the sets of morphisms:

$$\text{Hom}_{\underline{\text{Grp}}}(G, \mathcal{A}(H)) \cong \text{Hom}_{\underline{\text{Ab}}}(G^{ab}, H)$$

   Remark: Since this bijection is "natural" (a condition we won't formally define or check) it means that $\mathcal{A} : \underline{\text{Ab}} \to \underline{\text{Grp}}$ and $ab : \underline{\text{Grp}} \to \underline{\text{Ab}}$ are what we call a pair of *adjoint functors*.

4. We proved in class that the map $\text{Hom}_R(D, -) : R\text{–}\underline{\text{Mod}} \to \underline{\text{Ab}}$ is a covariant functor.

(a) We have another name for the functor of abelian groups $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, -)$. What is it?

(b) To which groups does the functor $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, -)$ map the $\mathbb{Z}$–modules $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$, $(\mathbb{Z}/n\mathbb{Z})^p$, $\mathbb{Z}/n^p\mathbb{Z}$, and $\mathbb{Z}/m\mathbb{Z}$ (for m, n coprime)? Express your solutions as a product of cyclic groups. You can simply list the answers; no proof needed.

(c) Describe the sequence of abelian groups and the maps obtained by applying $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, -)$ to the following short exact sequences:

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\text{mod } 2} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\psi} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \xrightarrow{\varphi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{\text{mod } 2} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

(Here, ψ is the inclusion of the first direct summand $\mathbb{Z}/2\mathbb{Z}$ and φ is the projection map onto the second direct summand $\mathbb{Z}/2\mathbb{Z}$.)

In particular, state which resulting sequence is exact.

5. **Bonus (Optional).** Let I be a (possibly infinite) index set, and let M_i, $(i \in I)$, and N be R–modules.

(a) (i) Prove the following isomorphisms of abelian groups: $\mathrm{Hom}_R\left(\bigoplus_{i\in I} M_i,\ N\right) \cong \prod_{i\in I} \mathrm{Hom}_R(M_i, N)$

(ii) Use this isomorphism to state a universal property for the direct sum $\bigoplus_{i\in I} M_i$.

(b) (i) Prove the following isomorphisms of abelian groups: $\mathrm{Hom}_R\left(N, \prod_{i\in I} M_i\right) \cong \prod_{i\in I} \mathrm{Hom}_R(N, M_i)$

(ii) Use this isomorphism to state a universal property for the direct product $\prod_{i\in I} M_i$.

6. **Bonus (Optional).** Show that a $\mathbb{Z}$–linearly independent subset B of the free abelian group $\mathbb{Z}^N$ can be extended to a basis for $\mathbb{Z}^N$ if and only if $\mathbb{Z}B$ is *splittable* in the sense of Homework #2 Bonus.