Reading: Dummit–Foote Ch 10.5.

Summary of definitions and main results

Definitions we've covered: split short exact sequence

Main results: Short Five Lemma, Splitting Lemma, equivalent definition of Noetherian ring

Warm-Up Questions

1. Use the Splitting Lemma to show that if m and n are coprime, the following short exact sequence splits:

$$0 \longrightarrow \mathbb{Z}/m\mathbb{Z} \xrightarrow{\psi} \mathbb{Z}/mn\mathbb{Z} \xrightarrow{\phi} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0.$$

   What if m and n are not coprime?

2. An R–submodule N of an R-module M has a *direct complement* P if $M \cong N \oplus P$ for P an R–submodule.
   (a) Show that the $\mathbb{Z}$–submodule $2\mathbb{Z} \subseteq \mathbb{Z}$ does not have a direct complement.
   (b) Show that the $\mathbb{Z}$–submodule $(3) \subseteq \mathbb{Z}/9\mathbb{Z}$ does not have a direct complement.
   (c) Show that the $\mathbb{Z}$–submodule $(3) \subseteq \mathbb{Z}/6\mathbb{Z}$ *does* have a direct complement.
   (d) Let V be the the $\mathbb{Q}[x]$–module where x acts by the matrix $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. Show that $U = \text{span}_{\mathbb{Q}}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ is a $\mathbb{Q}[x]$–submodule of V, and that it has a direct complement.
   (e) Let V be the the $\mathbb{Q}[x]$–module where x acts by the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Show that $U = \text{span}_{\mathbb{Q}}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ is a $\mathbb{Q}[x]$–submodule of V, and that it has no direct complement.
   (f) Show that the $\mathbb{Q}[x]$–submodule $(x) \subseteq \mathbb{Q}[x]/(x^3)$ does not have a direct complement.
   (g) Show that every linear subspace of a vector space has a direct complement.
   (h) Show that $A = \text{span}_{\mathbb{Z}}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ is a $\mathbb{Z}$–submodule of $\mathbb{Z}^2$ that has a direct complement.
   (i) Show that $B = \text{span}_{\mathbb{Z}}\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right)$ is a $\mathbb{Z}$–submodule of $\mathbb{Z}^2$ that does not have a direct complement.
   (j) Does $C = \text{span}_{\mathbb{Z}}\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) \subseteq \mathbb{Z}^2$ have a direct complement?

3. Re-interpret each of the parts of Problem (2) in terms of the splitting or non-splitting of a short exact sequence.

4. Let $0 \to A \to B \to C \to 0$ be an exact sequence of R–modules.
   (a) Show that if B is torsion, so are A and C.
   (b) Prove or find a counterexample: If A and C are torsion, then so is B.
   (c) Show by example that if B is torsion-free, then A is torsion-free, but C need not be.
   (d) Prove or find a counterexample: If A and C are torsion-free, then so is B.

Assignment Questions

1. **(Short Five Lemma).** Consider a homomorphism of short exact sequences of R–modules:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\psi} & B & \xrightarrow{\varphi} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & A' & \xrightarrow[\psi']{} & B' & \xrightarrow[\varphi']{} & C' & \longrightarrow & 0 \end{array}$$

   Prove the remaining step in the Short Five Lemma: If α and γ both surject, then β must also surject.

2. **(The Splitting Lemma).** Let R be a ring, and consider the short exact sequence of R–modules:

$$0 \longrightarrow A \xrightarrow{\psi} B \xrightarrow{\varphi} C \longrightarrow 0.$$

   Prove that the following are equivalent.

   (i) The sequence *splits*, that is, B is isomorphic to $A \oplus C$ such that ψ corresponds to the natural inclusion of A, and φ corresponds to the natural projection onto C.

   (ii) There is a map $\varphi' : C \to B$ such that $\varphi \circ \varphi'$ is the identity on C.

$$0 \longrightarrow A \xrightarrow{\psi} B \underset{\varphi'}{\overset{\varphi}{\rightleftarrows}} C \to 0$$

   (iii) There is a map $\psi' : B \to A$ such that $\psi' \circ \psi$ is the identity on A.

$$0 \to A \underset{\psi'}{\overset{\psi}{\rightleftarrows}} B \xrightarrow{\varphi} C \longrightarrow 0$$

   The maps φ' and ψ' are called *splitting homomorphisms.*

3. Recall that a ring R is defined to be *(left) Noetherian* if R is Noetherian as a left module over itself, that is, every R–submodule of R is finitely generated. In this question we will show this definition is equivalent to the following alternate definition of a Noetherian ring: R is *(left) Noetherian* if **every** finitely generated (left) R–module is Noetherian.

   (a) Let $0 \to A \to B \to C \to 0$ be a short exact sequence of R–modules. Show that if A and C are finitely generated R–modules, then B is finitely generated.

   (b) Let M be a submodule of R^n. Consider the short exact sequence of R–modules

$$0 \longrightarrow \{0\} \times R^{n-1} \longrightarrow R^n \xrightarrow{\pi_1} R \longrightarrow 0,$$

   (Here, π_1 is the projection onto the first factor of R^n.) Show that we obtain a short exact sequence

$$0 \longrightarrow M \cap \left(\{0\} \times R^{n-1}\right) \longrightarrow M \longrightarrow \pi_1(M) \longrightarrow 0.$$

   (c) Suppose R is Noetherian as a left R–module. Using parts (a) and (b) and induction on n, prove that R^n is a Noetherian R–module.

   (d) Prove that an R–module N is finitely generated if and only if it is quotient of a finite rank free R–module R^n.

   (e) Prove that a quotient of a Noetherian R–module is Noetherian.

   (f) Conclude that if R is Noetherian as a left R–module, then any finitely generated R–module is Noetherian.