Reading: Dummit–Foote Ch 10.4.

Summary of definitions and main results

Definitions we've covered: exact functor, left exact functor, right exact functor, tensor product of modules (as an abelian group), (S, R)–bimodule, tensor product of modules (as an S–module), R–balanced map, R–bilinear map, universal property of the tensor product (version for abelian groups, and version for R–module structure when R is commutative), tensor product of maps, extension of scalars, R–algebra, Jordan block with eigenvalue λ.

Main results: $\text{Hom}_R(D, -)$ is left exact, $\text{Hom}_R(-, D)$ is a contravariant functor and is left exact, an explicit construction of the tensor product $M \otimes_R N$, verification that the construction satisfies the universal property, tensor product distributes over direct sums, $R^n \otimes_R N \cong N^n$, tensor product is associative, using the universal property to perform computations

Warm-Up Questions

1. Let R be a ring, let A be a right R–module and B a left R–module. Prove that the universal property of the tensor product defines the abelian group $A \otimes_R B$ uniquely up to unique isomorphism.

2. Let R be a ring with right R–module M and left R–module N. What is the additive identity in the tensor product $M \otimes_R N$? Explain why the elements $m \otimes 0$ and $0 \otimes n$ are zero for any $m \in M$ or $n \in N$.

3. (a) Explain why, when R is commutative, a left R–module M will also be a right R–module under the action $mr = rm$, and conversely any right R–module N has an induced left R–module structure.
   (b) Verify that these actions give M the structure of an (R, R)–bimodule (simply called an *R–bimodule*).
   (c) Why will these constructions generally not work when R is non-commutative?

4. Let R be a ring with right R–module M and left R–module N. Show that the natural map
$$M \times N \longrightarrow M \otimes_R N$$
is **not** a group homomorphism in general. What are the constraints on this map, as imposed by the defining relations of $M \otimes_R N$?

5. (a) Let R and S be rings (possibly the same ring). Let M be a right R–module and N a left R–module. When will the tensor product $M \otimes_R N$ have the structure of an abelian group, and under what conditions will it additionally have the structure of an S–module?
   (b) Verify that if M is an (S, R)–bimodule, then $M \otimes_R N$ has an S–module structure. Why do we need to assume that the actions of S and R on M commute?
   (c) Suppose that M is an (S, R)–bimodule. Find a way to modify the statement of the universal property of the tensor product so that it determines $M \otimes_R N$ as an S–module (and not just as an abelian group).

6. Let R be a commutative ring. Let e_1, e_2, e_3 be a basis for the R^3 and let f_1, f_2, f_3, f_4 be a basis for R^4. Expand the tensor
$$(a_1e_1 + a_2e_2 + a_3e_3) \otimes (b_1f_1 + b_2f_2 + b_3f_3 + b_4f_4) \qquad \in R^3 \otimes_R R^4.$$

7. Let R be a ring with right R–module M and left R–module N. Which of the following maps are R–balanced? Which are homomorphisms of abelian groups? For the maps that are R–balanced, describe how they factor through the tensor product.

(a) The identity map $M \times N \longrightarrow M \times N$.

(b) The natural projections of $M \times N$ onto M and N.

(c) The natural map $M \times N \longrightarrow M \otimes_R N$.

(d) Suppose M and N are ideals of R. The multiplication map

$$\begin{aligned} M \times N &\longrightarrow R \\ (m,n) &\longmapsto mn \end{aligned}$$

(e) Suppose R is commutative. The matrix multiplication map

$$\begin{aligned} M_{n\times k}(R) \times M_{k\times m}(R) &\longrightarrow M_{n\times m}(R) \\ (A,B) &\longmapsto AB \end{aligned}$$

(f) Suppose R is commutative and M, N, P are R–modules. The composition map:

$$\begin{aligned} \mathrm{Hom}_R(M,N) \times \mathrm{Hom}_R(N,P) &\longrightarrow \mathrm{Hom}_R(M,P) \\ (f,g) &\longmapsto g \circ f \end{aligned}$$

(g) Suppose R is commutative. The dot product map:

$$\begin{aligned} R^n \times R^n &\longrightarrow R \\ (v,w) &\longmapsto v \cdot w \end{aligned}$$

(h) Suppose R is commutative. The cross product map:

$$\begin{aligned} R^3 \times R^3 &\longrightarrow R^3 \\ (v,w) &\longmapsto v \times w \end{aligned}$$

(i) Suppose R is commutative. The determinant map:

$$\begin{aligned} R^2 \times R^2 &\longrightarrow R \\ (v,w) &\longmapsto \det \begin{bmatrix} | & | \\ v & w \\ | & | \end{bmatrix} \end{aligned}$$

8. Let $V \cong \mathbb{C}^2$ be a complex vector space, and let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix with respect to the standard basis e_1, e_2. Write down the matrix for the linear map induced by A on the four-dimensional vector space $V \otimes V$ with respect to the basis $e_1 \otimes e_1,\ e_1 \otimes e_2,\ e_2 \otimes e_1,\ e_2 \otimes e_2$.

9. Let V be a complex vector space. Let $T : V \to V$ be a diagonalizable linear map with eigenbasis $v_1, v_2, \dots v_n$, and associated eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. What are the eigenvalues of the map induced by T on $V \otimes V$, and what are the associated eigenvectors?

10. Let R be a ring and S a subring.
    (a) Give an example of R, S and an S–module that embeds into an R–module.
    (b) Give an example of R, S, and an S–module that cannot embed into any R–module.

11. (a) We defined how to form the tensor product $M \otimes_R N$ of a right R–module M and a left R–module N. What would go wrong with this construction if M instead had the structure of a left R–module?
    (b) Show that if M is an (S,R)–bimodule and N a left R–module, the tensor product $M \otimes_R N$ has the structure of an S–module. Why must the left action of S and the right action of R on M commute?

12. Verify that the tensor product of maps respects composition:

$$(\tilde{\phi} \otimes \tilde{\psi}) \circ (\phi \otimes \psi) = (\tilde{\phi} \circ \phi) \otimes (\tilde{\psi} \circ \psi).$$

13. Fill in the details of the proof of that the tensor product associates (Dummit–Foote 10.4 Theorem 14).

Assignment Questions

1. **(The functor** $\mathrm{Hom}_R(-, D)$**).**

   (a) Show that if D is any R–module, then there is a **contravariant** functor

   $$\begin{aligned} \mathrm{Hom}_R(-, D) : R\text{–}\underline{\mathrm{Mod}} &\longrightarrow \underline{\mathrm{Ab}} \\ M &\longmapsto \mathrm{Hom}_R(M, D) \\ [\phi : M \to N] &\longmapsto \left[\begin{smallmatrix} \phi^* : \mathrm{Hom}_R(N,D) \longrightarrow \mathrm{Hom}_R(M,D) \\ f \longmapsto f\circ\phi \end{smallmatrix}\right] \end{aligned}$$

   Remark: The vector space dual functor $\mathrm{Hom}_k(-, k) : V \mapsto \mathrm{Hom}_k(V, k) = V^*$ is a special case.

   (b) Show that $\mathrm{Hom}_R(-, D)$ is left exact. This means (for a contravariant functor) that for any short exact sequence

   $$0 \longrightarrow A \xrightarrow{\psi} B \xrightarrow{\phi} C \to 0$$

   the following is exact:

   $$0 \longrightarrow \mathrm{Hom}_R(C, D) \xrightarrow{\phi^*} \mathrm{Hom}_R(B, D) \xrightarrow{\psi^*} \mathrm{Hom}_R(A, D).$$

   (c) Show that this functor may not be exact by finding an example of an injective R-module map $\psi : M \to N$ such that ψ^* is not surjective.

2. Let R be a **commutative** ring. Let M, N, L be R–modules.

   (a) Show that any R–module has the structure of an (R, R)–bimodule.

   (b) Verify that the tensor product $M \otimes_R N$ (as defined by the quotient of abelian groups $F(M \times N)/I_\otimes$) has a well-defined R–module structure.

   (c) A map $\varphi : M \times N \to L$ is called R*–bilinear* if it satisfies the following properties:

   $$\begin{aligned} \varphi(r_1m_1 + r_2m_2, n) &= r_1\varphi(m_1, n) + r_2\varphi(m_2, n) \qquad (R\text{–linearity in } M) \\ \varphi(m, r_1n_1 + r_2n_2) &= r_1\varphi(m, n_1) + r_2\varphi(m, n_2) \qquad (R\text{–linearity in } N) \\ &\text{for all } m, m_1, m_2 \in M, n, n_1, n_2 \in N, r_1, r_2 \in R. \end{aligned}$$

   Prove that the tensor product $M \otimes_R N$ (viewed as an R–module) satisfies the following universal property. For any R–bilinear map $\varphi : M \times N \to L$ there is a unique R–linear map $\Phi : M \otimes_R N \to L$ that makes the following diagram commute.

   $$\begin{array}{ccc} M \times N & \longrightarrow & M \otimes_R N \\ & \searrow^{\varphi} & \Big\downarrow {\scriptstyle \exists!\Phi} \\ & & L \end{array}$$

   (d) Conclude that there is a bijection between the set of R–bilinear maps $M \times N \to L$ and the set of R–linear maps $M \otimes_R N \to L$.

3. (a) Let M be a right R–module and $N_1, \ldots, N_k$ a set of left R–modules. Prove that the tensor product distributes over direct sums, that is, prove that there is an isomorphism of groups defined by the map

   $$\begin{aligned} M \otimes_R (N_1 \oplus \cdots \oplus N_k) &\xrightarrow{\cong} (M \otimes_R N_1) \oplus \cdots \oplus (M \otimes_R N_k) \\ m \otimes (n_1, \ldots, n_k) &\longmapsto (m \otimes n_1, m \otimes n_2, \ldots, m \otimes n_k) \end{aligned}$$

Verify that if M is an (S, R)–bimodule, then this is an isomorphism of S–modules.
Hint: Dummit–Foote Chapter 10.4 Theorem 17.
Remark: A similar argument (which you need not verify) shows that

$$(M_1 \oplus \cdots \oplus M_k) \otimes_R N \xrightarrow{\cong} (M_1 \otimes_R N) \oplus \cdots \oplus (M_k \otimes_R N).$$

(b) Conclude that if N is a left R–module, $R^k \otimes_R N \cong N^k$.

(c) Show that if $F(A) \cong \bigoplus_A R$ is the free R–module on a set A, and $F(B) \cong \bigoplus_B R$ is the free R–module on the set B, then $F(A) \otimes_R F(B)$ is the free R–module on the set $A \times B$.
Remark: In particular, this proves our claim about tensor products of vector spaces!

(d) Show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ are **not** isomorphic as vector spaces over $\mathbb{R}$.

4. (a) Let D be a right R–module. Prove that this map is a (well-defined) additive covariant functor:

$$\begin{aligned} D \otimes_R - : R\text{–}\underline{\text{Mod}} &\longrightarrow \underline{\text{Ab}} \\ M &\longmapsto D \otimes_R M \\ [\phi : M \to N] &\longmapsto \begin{bmatrix} \phi_* : D \otimes_R M \longrightarrow D \otimes_R N \\ \phi_*(d \otimes m) = d \otimes \phi(m) \end{bmatrix} \end{aligned}$$

(b) Let k be a field, and V a k-vector space. Prove that the functor $V \otimes_k - : k\text{–}\underline{\text{Vect}} \longrightarrow k\text{–}\underline{\text{Vect}}$ is exact.

5. **(Building toward a theory of Jordan Canonical Form: Part 1).** Let V be the $\mathbb{C}[x]$–module

$$V = \frac{\mathbb{C}[x]}{(x - \lambda)^k} \qquad \text{for some } \lambda \in \mathbb{C} \text{ and } k \in \mathbb{Z}_{>0}.$$

We write T to denote the $\mathbb{C}$–linear map $T : V \to V$ defined by multiplication by $x \in \mathbb{C}[x]$.

(a) Show that $\{(x - \lambda)^{k-1}, (x - \lambda)^{k-2}, \cdots, (x - \lambda), 1\}$ is a basis for V.

(b) Write down the matrix for the action of T on V with respect to the ordered basis

$$(x - \lambda)^{k-1}, \quad (x - \lambda)^{k-2}, \quad \cdots \quad (x - \lambda), \quad 1.$$

A matrix of this form is called a *Jordan block* with eigenvalue λ.

(c) Verify that λ is the only eigenvalue of this Jordan block, and that the associated eigenspace is 1–dimensional.

6. **Bonus (Optional). (The functor $D \otimes_R -$ is right exact.)** Let R be any ring, and D a right R–module.

(a) Show by example that the functor $D \otimes_R - : R\text{–}\underline{\text{Mod}} \to \mathbb{Z}\text{–}\underline{\text{Mod}}$ may *not* be an exact functor.

(b) Show that $D \otimes_R -$ is right exact. *Hint:* Dummit–Foote 10.5 Theorem 39.

7. **Bonus (Optional).**

(a) Let R be a ring and N a left R–module. Let $I \subseteq R$ be an ideal. Define the R–submodule $IN = \{a_1 n_1 + \cdots + a_k n_k \mid a_i \in I, n_i \in N\}$. Prove that

$$R/I \otimes_R N \cong N/IN$$

(i) using the universal property of the tensor product to construct an R–linear map and a two-sided inverse,

(ii) using the presentation

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

for R/I and the fact that the tensor product is right-exact.

(b) Use this result to compute the abelian group $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ (as a product of cyclic groups) for any $n, m \in \mathbb{Z}$.

(c) Let k be a field and let $R = k[x, y]$. Give simple descriptions of the following R–modules, and determine their dimensions over k.

$$\frac{R}{(x)} \otimes_R \frac{R}{(x-y)} \qquad \frac{R}{(x)} \otimes_R \frac{R}{(x-1)} \qquad \frac{R}{(y-1)} \otimes_R \frac{R}{(x-y)}$$