Reading: Dummit–Foote Ch 11.5, 18.1, Fulton–Harris Ch 1.1–1.2.

Summary of definitions and main results

Definitions we've covered: k^{th} tensor power $T^k(M)$, tensor algebra $T^*(M)$, k^{th} symmetric power $\mathrm{Sym}^k(M)$, symmetric algebra $\mathrm{Sym}^*(M)$, k^{th} exterior power $\bigwedge^k M$, exterior algebra $\bigwedge^* M$, group ring, (linear) representation, degree of a representation, faithful representation, trivial representation, permutation representation, regular representation, homomorphism and isomorphism of representations, G-equivariant map, intertwiner, minimal polynomial of a linear map

Main results: using right exactness to compute tensor products, construction & universal properties for tensor, symmetric, and exterior powers and algebras, equivalent definitions of a group representation

Warm-Up Questions

1. Show that the following alternate definition of an R–algebra A is equivalent to the one from class. Given a commutative ring R, an R*–algebra* A is an R–module A with a ring structure such that the multiplication map $A \times A \to A$ is R–bilinear.
2. Let R be a commutative ring, and M and R–module.
   (a) Verify that, if 2 is invertible in R, then the submodule
   $$\langle m_1 \otimes m_2 \otimes \cdots \otimes m_k \mid m_i = m_j \text{ for some } i \neq j \rangle \subseteq T^k M$$
   defining the exterior power $\bigwedge^k M$ is equal to the submodule
   $$\langle m_1 \otimes m_2 \otimes \cdots \otimes m_k - \mathrm{sign}(\sigma) m_{\sigma(1)} \otimes m_{\sigma(2)} \otimes \cdots \otimes m_{\sigma(k)} \mid \sigma \in S_k \rangle.$$
   (b) Are these submodules the same when 2 is not invertible?
3. Let R be a commutative ring and M and R–module. Verify the universal properties for the R–modules
   $$\text{(a) } \mathrm{T}^k(M) \qquad \text{(b) } \mathrm{Sym}^k(M) \qquad \text{(c) } \bigwedge^k(M)$$
   and for R–algebras
   $$\text{(d) } \mathrm{T}^*(M) \qquad \text{(e) } \mathrm{Sym}^*(M)$$
4. Let G be a group and V an $\mathbb{F}$-vector space. Show that the following are all equivalent ways to define a (linear) representation of G on V.
   i. A group homomorphism $G \to \mathrm{GL}(V)$.
   ii. A group action (by linear maps) of G on V.
   iii. An $\mathbb{F}[G]$–module structure on V.
5. Let R be a commutative ring. Show that the group ring $R[\mathbb{Z}] \cong R[t, t^{-1}]$. Show that $R[\mathbb{Z}/n\mathbb{Z}] \cong R[t]/\langle t^n - 1\rangle$.What is the group ring $R[\mathbb{Z}^n]$? The group ring $R[\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}]$?
6. Let $\phi : G \to \mathrm{GL}(V)$ be any group representation. What is the image of the identity element in $\mathrm{GL}(V)$?
7. Compute the sum and product of $(1 + 3e_{(1\,2)} + 4e_{(1\,2\,3)})$ and $(4 + 2e_{(1\,2)} + 4e_{(1\,3)})$ in the group ring $\mathbb{Q}[S_3]$.
8. Let G be a group and R a commutative ring. Show that $R[G]$ is commutative if and only if G is abelian.
9. Given any representation $\phi : G \to \mathrm{GL}(V)$, prove that ϕ defines a faithful representation of $G/\ker(\phi)$.

10. (a) Find an explicit isomorphism T between the following two representations of S_2.

$$S_2 \longrightarrow \mathrm{GL}(\mathbb{R}^2) \qquad\qquad S_2 \longrightarrow \mathrm{GL}(\mathbb{R}^2)$$
$$(1\ 2) \longmapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad\qquad (1\ 2) \longmapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Give a geometric description of the action and the bases for $\mathbb{R}^2$ associated to each matrix group.

(b) Prove that the following two representations of S_2 are not isomorphic.

$$S_2 \longrightarrow \mathrm{GL}(\mathbb{R}^2) \qquad\qquad S_2 \longrightarrow \mathrm{GL}(\mathbb{R}^2)$$
$$(1\ 2) \longmapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad\qquad (1\ 2) \longmapsto \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

11. **(Linear algebra review.)** Let A, B, C be linear maps $V \to V$, with C invertible. Verify the following properties of the trace.

(a) $\mathrm{Trace}(CAC^{-1}) = \mathrm{Trace}(A)$ (so trace does not depend on choice of basis or matrix representing A).

(b) $\mathrm{Trace}(cA + B) = c\mathrm{Trace}(A) + \mathrm{Trace}(B)$ for any scalar c.

(c) $\mathrm{Trace}(AB) = \mathrm{Trace}(BA)$ but $\mathrm{Trace}(AB) \neq \mathrm{Trace}(A)\mathrm{Trace}(B)$ in general.

(d) $\mathrm{Trace}(A) = \mathrm{Trace}(A^T)$.

(e) $\mathrm{Trace}(\mathrm{Id}_V) = \dim(V)$.

(f) $\mathrm{Trace}(A)$ is the sum of the eigenvalues of A (with algebraic multiplicity).

(g) If A has characteristic polynomial $p_A(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then $\mathrm{Trace}(A) = a_{n-1}$.

(h) If $V = U \oplus W$ and U, W are stabilized by A, then $\mathrm{Trace}(A) = \mathrm{Trace}(A|_U) + \mathrm{Trace}(A|_W)$.

12. **(Linear algebra review.)**

(a) Define what it means for two matrices to be *conjugate* (or *similar*)

(b) What is the conjugacy class of the zero matrix? The identity matrix? A scalar matrix?

(c) Explain why two matrices are conjugate if and only if they represent the same linear map with respect to different bases.

(d) Show that conjugate matrices have the same determinant.

(e) Show that $(ABA^{-1})^n = AB^nA^{-1}$.

13. **(Linear algebra review.)** Let $A : V \to V$ be a linear map on a finite dimensional vector space V.

(a) Suppose A is a *block diagonal matrix*, ie, it has square matrices $\mathbf{A}_i$ (its *blocks*) on the diagonal:

$$A = \begin{bmatrix} \mathbf{A}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{A}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{A}_n \end{bmatrix} \qquad \left(\text{eg. } \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 5 & 0 \\ 0 & 3 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \text{ has } \mathbf{A}_1 = [1], \mathbf{A}_2 = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}, \mathbf{A}_3 = [4] \right)$$

Explain how the blocks of A correspond to a decomposition of V into a direct sum of subspaces $V = V_1 \oplus \cdots \oplus V_n$ where each V_i is invariant under the action of A. (The matrix A is sometimes called the *direct sum* of its blocks $A = \mathbf{A}_1 \oplus \mathbf{A}_2 \oplus \cdots \oplus \mathbf{A}_n$.)

(b) Conversely, explain why, if V decomposes into a direct sum of subspaces that are invariant under A, then the corresponding matrix for A will be block diagonal. (What are the sizes of the blocks?)

(c) Observe that $\mathrm{Trace}(A) = \mathrm{Trace}(\mathbf{A}_1) + \cdots + \mathrm{Trace}(\mathbf{A}_n)$, and $\mathrm{Det}(A) = \mathrm{Det}(\mathbf{A}_1) \cdots \mathrm{Det}(\mathbf{A}_n)$.

(d) What is the product of two block diagonal matrices (assuming blocks of the same sizes)?

(e) Show that for any exponent $p \in \mathbb{Z}_{\geq 0}$, the matrix A^p is block diagonal with blocks $\mathbf{A}_1^p, \ldots, \mathbf{A}_m^p$.

Assignment Questions

For this assignment, you may quote basic results from linear algebra (including facts about matrix inverses, transpose, trace, and determinant) and basic facts about complex conjugation without proof.

1. Let R be a commutative ring and M an R–module.
   (a) For any commutative ring R and R–module M, show that the R–module $T^*M := \bigoplus_{i=0}^{\infty} M^{\otimes i}$ has the structure of an R–algebra.
   (b) A similar proof shows that $\mathrm{Sym}^*M := \bigoplus_{i=0}^{\infty} \mathrm{Sym}^i(M)$ and $\bigwedge^* M := \bigoplus_{i=0}^{\infty} \bigwedge^i M$ are R–algebras. You do not need to give a full proof, but verify that multiplication is well-defined for these spaces (it is independent of representative of an equivalence class of elements in these quotients).

2. Let $\mathbb{F}$ be a field of characteristic zero and V a vector space over $\mathbb{F}$ with basis $\{x_1, \ldots, x_n\}$.
   (a) Let W be any vector space over $\mathbb{F}$, and $v_1, \ldots, v_N$ elements of W. Prove that, to show that the elements v_i are linearly independent, it suffices to construct $\mathbb{F}$–linear maps $\phi_i : W \to \mathbb{F}$ such that

$$\phi_i(v_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

   (b) Verify that $\mathrm{Sym}^k(V)$ is a vector space over $\mathbb{F}$ with basis given by the set of monomials in the variables $\{x_1, x_2, \ldots, x_n\}$ of total degree k. (*Remark:* There are $\binom{n+k-1}{n-1}$ such monomials).
   (c) Verify that $\bigwedge^k V$ is isomorphic to the $\mathbb{F}$–vector space with a basis given by elements of the form $x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_k}$ with $i_1 < i_2 < \cdots < i_k$. (*Remark:* There are $\binom{n}{k}$ such elements).
   (d) Suppose that $A : V \to V$ is a diagonalizable linear map with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (listed with multiplicity). Compute the eigenvalues of the maps induced by A on T^kV, $\mathrm{Sym}^k(V)$, and $\wedge^k V$.
   (e) Show that you can identify Sym^*V, and $\bigwedge^* V$ as **direct summands** of T^*V via the (split) maps

$$x_1x_2\cdots x_k \longmapsto \frac{1}{k!}\sum_{\sigma \in S_k} \sigma(x_1 \otimes x_2 \otimes \cdots \otimes x_k) \quad \text{and} \quad x_1 \wedge x_2 \wedge \cdots \wedge x_k \longmapsto \frac{1}{k!}\sum_{\sigma \in S_k} \mathrm{sign}(\sigma)\sigma(x_1 \otimes x_2 \otimes \cdots \otimes x_k)$$

   (We are using the assumption that $\mathbb{F}$ has characteristic zero, so the integer $k!$ is invertible in $\mathbb{F}$.)
   (f) Show that $V \otimes_{\mathbb{F}} V \cong \mathrm{Sym}^2(V) \oplus \wedge^2 V$.
   Remark: If V has dimension at least 2, then $V \otimes_{\mathbb{F}} V \otimes_{\mathbb{F}} V \supsetneq \mathrm{Sym}^3(V) \oplus \wedge^3 V$.

3. **(Building toward a theory of Jordan Canonical Form: Part 2).** Let V be a $\mathbb{C}[x]$–module that is finite dimensional over $\mathbb{C}$, where x acts on V by a $\mathbb{C}$–linear map T. According to the structure theorem for finitely generated modules over a PID, we can write

$$V \cong \frac{\mathbb{C}[x]}{(p_1(x))} \oplus \frac{\mathbb{C}[x]}{(p_2(x))} \oplus \cdots \oplus \frac{\mathbb{C}[x]}{(p_k(x))}$$

   for some monic polynomials $p_i(x) \in \mathbb{C}[x]$ such that $p_1(x)$ divides $p_2(x)$, $p_2(x)$ divides $p_3(x)$, etc.

   The monic polynomial $p_k(x)$ is called the *minimal polynomial* of T, and the product $p_1(x)p_2(x)\cdots p_k(x)$ is called the *characteristic polynomial* of T. By construction the minimal and characteristic polynomials have the same set of roots (possibly with different multiplicities).

   (a) Briefly explain why V can also be further decomposed as a direct sum

$$V \cong \frac{\mathbb{C}[x]}{(x-\lambda_1)^{k_1}} \oplus \frac{\mathbb{C}[x]}{(x-\lambda_2)^{k_2}} \oplus \cdots \oplus \frac{\mathbb{C}[x]}{(x-\lambda_d)^{k_d}}$$

   for (not necessarily distinct) scalars $\lambda_i \in \mathbb{C}$ and positive powers k_i. Explain the relationship between the scalars λ_i, the mulitplicities k_i, and the polynomials $p_j(x)$. *Hint:* Chinese Remainder Theorem.

(b) Conclude that the matrix T can be expressed as a block diagonal matrix, where each block is a Jordan block. This is called the *Jordan canonical form* of T. *Hint:* Homework 6 Question #5.

(c) Suppose that $\mu \in \mathbb{C}$ is *not* a root of $p_k(x)$ (and therefore not a root of $p_j(x)$ for any j). Show that μ is not an eigenvalue of T. Conclude that the eigenvalues of T are precisely the roots of the minimal polynomial $p_k(x)$.

Hint: Consider the projection of a μ–eigenspace onto the summand $\dfrac{\mathbb{C}[x]}{(x-\lambda_i)^{k_i}}$ for each i.

(d) Show that $\text{Ann}(V) = \big(p_k(x)\big)$.

(e) Show that $\text{Ann}(V)$ is equal to the set

$$\{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{C}[x] \mid a_n T^n + a_{n-1}T^{n-1} + \cdots + a_1 T + a_0 I \text{ is the zero map}\}$$

(f) Conclude that if $p(T) = 0$ for some polynomial $p(x)$, every eigenvalue of T is a root of $p(x)$.

(g) Show that T is diagonalizable if and only if the roots of its minimal polynomial $p_k(x)$ are distinct, ie, they each occur with multiplicity one.

(h) **(Application to representation theory.)** Suppose the linear map T has finite order, that is, $T^n = I$ for some $n \in \mathbb{Z}_{\geq 0}$. Show that T is diagonalizable, and that every eigenvalue is an n^{th} root of unity. Use this result to conclude the following fact about complex representations of finite groups: Let G be a finite group of order n, and let $\rho : G \to \text{GL}(V)$ be a representation of G on a finite dimensional $\mathbb{C}$–vector space V. For every $g \in G$ the linear map $\rho(g)$ is diagonalizable, and its eigenvalues are n^{th} roots of unity.

4. Let G be a finite group, and $\mathbb{F}$ a field. You may use properties of the trace without proof.

(a) Let $G \to \text{GL}(U)$ be any representation of G. Citing facts from linear algebra (which you don't need to prove), explain why the trace of the matrix representing a given element $g \in G$ is well-defined in the sense that it will be the same in any isomorphic representation of G.

(b) A *permutation representation* of G on a finite-dimensional $\mathbb{F}$-vector space V is a linear representation $\rho : G \to \text{GL}(V)$ in which elements act by permuting some basis $B = \{b_1, \ldots b_m\}$ for V. Show that, with respect to the basis $\{b_1, \ldots, b_m\}$, for each element $g \in G$, $\rho(g)$ is represented by an $m \times m$ *permutation matrix*, a square matrix that has exactly one entry 1 in each row and each column, and zero elsewhere. Use this description of matrices $\rho(g)$ to show that the trace of $\rho(g)$ is equal to the number of basis elements b_i fixed by $\rho(g)$.

(c) Our first example of a permutation representation was given by the action of S_n on $\mathbb{F}^n$ by permuting the basis $e_1, \ldots, e_n$. Show, in contrast, that the subrepresentation

$$U = \{a_1 e_1 + a_2 e_2 + \cdots + a_n e_n \mid a_1 + a_2 + \cdots + a_n = 0\} \subseteq \mathbb{F}^n$$

is *not* a permutation representation with respect to any basis for U.
Hint: Warm-up Question 11(h). What is the trace of an n–cycle?

(d) The group ring of $\mathbb{F}[G]$ is a left module over itself. This corresponds to permutation representation of the group G on the underlying vector space $\mathbb{F}[G]$, called the *(left) regular representation* of G. Find the degree of this representation. In what basis is this a permutation representation, and how many G-orbits does this basis have?

(e) For any $g \in G$, compute the trace of the matrix representing g in the regular representation.

5. **(Bonus) (The tensor-Hom adjunction.)** Let S, R be rings. Let A be an (S, R)–bimodule, B a left R–module, and C a left S–module. Prove that there is a (well-defined) isomorphism of abelian groups

$$\text{Hom}_S(A \otimes_R B, C) \xrightarrow{\cong} \text{Hom}_R\big(B, \text{Hom}_S(A, C)\big)$$
$$\big[f : a \otimes b \longmapsto f(a \otimes b)\big] \longmapsto \Big[b \longmapsto [a \longmapsto f(a \otimes b)]\Big]$$

It turns out that this bijection is *natural*, so the functors $A \otimes_R -$ and $\text{Hom}_S(A, -)$ are adjoints.