Reading: Fulton–Harris Chapter 1–2.

Summary of definitions and main results

Definitions we've covered: simple (or irreducible) module, decomposable module, completely reducible module, V^G, isotypic component.

Main results: Schur's lemma; properties of the averaging map; Maschke's theorem; induced $\mathbb{F}[G]$–modules structures on $V \oplus W$, $\mathrm{Hom}_{\mathbb{F}}(V, W)$, V^*, $V \otimes W$, $\wedge^k V$, $\mathrm{Sym}^k(V)$, decomposition into irredcible representations is unique.

Warm-Up Questions

1. Given a group representation $\phi : G \to \mathrm{GL}(V)$ over a field $\mathbb{F}$, prove that the map
$$\begin{aligned} G &\longrightarrow \mathbb{F}^\times = \mathrm{GL}(\mathbb{F}) \\ g &\longmapsto \det(\phi(g)) \end{aligned}$$
defines a degree–1 representation of G.

2. Let G be a group generated by a set S. Suppose that $T : V \to W$ is a map of vector spaces bewteen G–representations V and W. Show that, to verify that T is G–equivariant, it suffices to check that $T(g \cdot v) = g \cdot T(v)$ for generators $g \in S$.

3. Given an example of a ring R and an R–module M that is:

   (a) irreducible

   (b) reducible, but not decomposable

   (c) decomposable, but not completely reducible

   (d) completely reducible, but not irreducible

4. Fix an integer $n > 0$. Recall the following example from class: The symmetric group S_n acts on $\mathbb{C}^n$ by permuting a basis $e_1, e_2, \ldots, e_n$. We saw that this representation has two subrepresentations,
$$D = \mathrm{span}_{\mathbb{C}}(e_1 + e_2 + \cdots e_n) \qquad \text{and} \qquad U = \{a_1 e_1 + a_2 e_2 + \cdots + a_n e_n \mid a_1 + a_2 + \cdots a_n = 0\}.$$
Show that, as a $\mathbb{C}S_n$–module, $\mathbb{C}^n$ is the direct sum $\mathbb{C}^n \cong D \oplus U$.

5. Let D_{2n} be the dihedral group, the symmetry group of a regular planar polygon with n edges. Draw the polygon in the plane $\mathbb{R}^2$ centred at the origin and with the y–axis as one of its lines of symmetry. Show that the action of D_{2n} on the polygon extends to a linear action of the plane. Verify that this is an irreducible degree–2 representation of D_{2n}.

6. Let V be a representation of a group G, and recall that V^G denotes the set of vectors in V that are fixed pointwise by the action of every group element $g \in G$. Verify that V^G is a linear subspace of V.

7. Complete our proof of Maschke's Theorem: Suppose $\pi_0 : V \to U$ is a projection map; this means $\pi_0(V) \subseteq U$ and $\pi_0|_U$ is the identity on U. Then the map
$$\pi = \frac{1}{|G|} \sum_{g \in G} g \pi_0 g^{-1} : V \to U$$
is also a projection.

8. (a) Let $\mathbb{C}^n = \langle e_1, \dots, e_n \rangle$ be the canonical representation of the symmetric group S_n by signed permutation matrices. Explicitly describe the action of the *averaging map* on $\mathbb{C}^n$:

$$\begin{aligned} \psi_{av} : \mathbb{C}^n &\longrightarrow \mathbb{C}^n \\ v &\longmapsto \frac{1}{n!} \sum_{\sigma \in S_n} \sigma \cdot v \end{aligned}$$

   (b) Suppose v is an element of the diagonal $D = \text{span}_{\mathbb{C}}(e_1 + e_2 + \cdots + e_n)$. What is $\psi_{av}(v)$?

   (c) Suppose v is an element of the standard subrepresentation $U = \{a_1 e_1 + \cdots + a_n e_n \mid \sum a_i = 0\}$. What is $\psi_{av}(v)$? *Hint:* First check $\psi_{av}(v)$ on the basis vectors $v = (e_1 - e_i)$ for U.

   (d) Interpret your answer to the previous questions, given that we know $\psi_{av} : V \to V$ is a linear projection onto V^G.

9. Let V and W be linear representations of a group G over a field $\mathbb{F}$.

   (a) Show that the tensor product $V \otimes_{\mathbb{F}} W$ has a (well-defined) induced diagonal action of G by

$$g \cdot (v \otimes w) = (g \cdot v) \otimes (g \cdot w).$$

   (b) Show that the tensor power $T^k(V)$ has an induced structure of a G–representation by the map

$$g \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = (g \cdot v_1) \otimes (g \cdot v_2) \otimes \cdots \otimes (g \cdot v_k)$$

   (c) Show that the above action decends to a well-defined action on $\text{Sym}^k(V)$ and $\bigwedge^k(V)$.

10. Suppose that V is a finite dimensional vector space over $\mathbb{F}$, and $T : V \to V$ is a diagonalizable linear map. Show that the restriction of T to any T–invariant subspace $W \subseteq V$ will also be diagonalizable, and therefore W must be a direct sum of eigenspaces of T.

11. Let G be a finite group. Show that an $\mathbb{F}[G]$–module V is finitely generated if and only if it is finite dimensional. What if G is infinite?

12. Let V, W be two representations of a group G, and let U_i be an irreducible G–representation. Let $T : V \to W$ be a G–equivariant map. Explain and prove the sense in which T must respect the isotypic component of U_i in V and W.

13. Let U and W be complex representations of a finite group G. Show that $(U \oplus W)^G \cong U^G \oplus W^G$.

14. Find the characteristic polynomial and the minimal polynomials of the following matrices.

$$\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \quad \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \quad \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix} \quad \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \quad \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

15. For each of the following $\mathbb{C}[x]$–modules, write the Jordan form of the linear map "multiplication by x". State the minimal and characteristic polynomials.

$$\frac{\mathbb{C}[x]}{(x-1)^2} \oplus \frac{\mathbb{C}[x]}{(x-1)(x-2)} \qquad \frac{\mathbb{C}[x]}{(x-1)(x-2)(x-3)} \qquad \frac{\mathbb{C}[x]}{(x-1)} \oplus \frac{\mathbb{C}[x]}{(x-1)^2} \oplus \frac{\mathbb{C}[x]}{(x-1)^2}$$

16. Determine all possible Jordan forms for linear maps with characteristic polynomial $(x-1)^3(x-2)^2$.

17. (a) Suppose a complex matrix A satisfies the equation $A^2 = -2A - 1$. What are the possibilities for its Jordan form?

   (b) Suppose a complex matrix A satisfies $A^3 = A$. Show that A is diagonalizable. Would this result hold if A had entries in a field of characteristic 2?

18. Prove that an $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Assignment Questions

For this assignment, you may quote basic results from linear algebra (including facts about matrix inverse, matrix conjugation, transpose, trace, and determinant) and basic facts about complex conjugation without proof.

Notation. Given a group representation $\varphi : G \to \mathrm{GL}(V)$ on a finite dimensional vector space V, for any $g \in G$, write $\chi_V(g)$ to denote the trace of the linear map $\varphi(g)$.

1. (a) Let G be a finite **abelian** group, and V a finite-dimensional complex representation of G. Show that the image of G in $GL(V)$ is *simultaneously diagonalizable*, that is, there is some basis for V with respect to which every matrix is diagonal. Conclude that V decomposes into a direct sum of 1-dimensional G–representations. *Hint:* Fulton–Harris Section 1.3.

   (b) It follows that all irreducible complex G–representations are 1–dimensional. Let C_n denote the cyclic group of order n. Find n irreducible degree–1 representations of C_n. Show that they are non–isomorphic, and comprise a complete list of its irreducible representations.

2. Let G be a group with linear actions on finite dimensional $\mathbb{F}$–vector spaces V and W, given by $\rho : G \to GL(V)$ and $\varphi : G \to \mathrm{GL}(W)$.

   (a) Show that the $\mathbb{F}$–vector space of linear maps $\mathrm{Hom}_{\mathbb{F}}(V, W)$ inherits the structure of a G–representation, where an element g in G acts by

   $$f \longmapsto \Big[v \mapsto \varphi(g) \Big(f\Big(\rho(g)^{-1}(v) \Big) \Big) \Big] \qquad \forall\ f \in \mathrm{Hom}_{\mathbb{F}}(V, W),\ g \in G$$

   (b) Consider the special case of this construction when $W = \mathbb{F}$ is the trivial G–representation, so $\mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F}) = V^*$. In this case, the induced representation of G on V^* is called the *dual representation* ρ^* of ρ, and given by

   $$\rho^*(g) : f \longmapsto \Big[v \mapsto f\Big(\rho(g)^{-1}(v) \Big) \Big] \qquad \forall\ f \in V^*,\ g \in G.$$

   If A is the matrix representing the action of a group element $g \in G$ on V with respect to a basis B, show that the matrix for g on V^* with respect to the dual basis B^* is given by $(A^{-1})^T$, the inverse transpose of A.

   (c) Now suppose that $k = \mathbb{C}$ and G is finite, and let $g \in G$. Prove that $\chi_{V^*}(g)$ is the complex conjugate of $\chi_V(g)$. *Hint:* What are its eigenvalues?

3. Let G be a finite group and $\mathbb{F}$ a field.

   (a) Let V and W be finite-dimensional representations of G over $\mathbb{F}$. Construct an isomorphism of $\mathbb{F}$–vector spaces $\mathrm{Hom}_{\mathbb{F}}(V, W) \cong V^* \otimes_{\mathbb{F}} W$. This isomorphism should be *natural*, that is, its definition should not require a choice of basis for V or W.
   (It's okay if you choose bases in the proof that it is an isomoprhism).

   (b) Suppose that $\mathbb{F}$ is algebraically closed. Suppose that A and B are finite order (therefore diagonalizable) endomorphisms of some finite dimensional vector spaces V and W over $\mathbb{F}$. Show that the trace of $A \otimes B$ on $V \otimes_{\mathbb{F}} W$ is the product $\mathrm{Trace}(A)\mathrm{Trace}(B)$.
   Remark: This result also holds when A and B are not diagonalizable, and can be proven (with a little more effort) by considering the bases for V and W putting A and B into Jordan canonical form. It can also be proven for arbitrary fields, using extension of scalars to the algebraic closure. See Question 6.

   (c) Let V and W be finite-dimensional representations of G over $\mathbb{C}$, and consider the induced action of G on $V \otimes_{\mathbb{C}} W$ defined on simple tensors by

   $$g \cdot (v \otimes w) = (g \cdot v) \otimes (g \cdot w).$$

Conclude that

$$\chi_{V\otimes_{\mathbb{C}}W}(g) = \chi_V(g)\chi_W(g) \qquad \text{for all } g \in G.$$

(d) Again let V and W be finite-dimensional representations of G over $\mathbb{C}$. Show that the isomorphism $\mathrm{Hom}_{\mathbb{C}}(V, W) \cong V^* \otimes_{\mathbb{C}} W$ is G–equivariant.

(e) Conclude that

$$\chi_{\mathrm{Hom}_{\mathbb{C}}(V,W)}(g) = \overline{\chi_V(g)}\chi_W(g) \qquad \text{for all } g \in G.$$

Remark: This will be a key result in our development of character theory!

4. **(Building toward a theory of Jordan Canonical Form: Part 3).** Let V be a $\mathbb{C}[x]$–module that is finite dimensional over $\mathbb{C}$. Recall that we have a decomposition

$$V \cong \frac{\mathbb{C}[x]}{(p_1(x))} \oplus \frac{\mathbb{C}[x]}{(p_2(x))} \oplus \cdots \oplus \frac{\mathbb{C}[x]}{(p_k(x))}$$

for some monic polynomials $p_i(x) \in \mathbb{C}[x]$ such that $p_1(x)$ divides $p_2(x)$, $p_2(x)$ divides $p_3(x)$, etc.

(a) The structure theorem for finitely generated modules over a PID implies that this decomposition is *unique*, that is, if

$$\frac{\mathbb{C}[x]}{(p_1(x))} \oplus \frac{\mathbb{C}[x]}{(p_2(x))} \oplus \cdots \oplus \frac{\mathbb{C}[x]}{(p_k(x))} \cong \frac{\mathbb{C}[x]}{(q_1(x))} \oplus \frac{\mathbb{C}[x]}{(q_2(x))} \oplus \cdots \oplus \frac{\mathbb{C}[x]}{(q_r(x))}$$

are two such decompositions then $r = k$ and $q_i(x) = p_i(x)$ for each i. Briefly conclude that the associated decomposition

$$V \cong \frac{\mathbb{C}[x]}{(x-\lambda_1)^{k_1}} \oplus \frac{\mathbb{C}[x]}{(x-\lambda_2)^{k_2}} \oplus \cdots \oplus \frac{\mathbb{C}[x]}{(x-\lambda_d)^{k_d}}.$$

is unique up to the order of the factors.

(b) Let U be a finite dimensional $\mathbb{C}$–vector space, and let $T : U \to U$ be a $\mathbb{C}$–linear map. Prove that the Jordan canonical form of T is *unique* in the sense that there is only one matrix in Jordan canonical form that represents the linear map T (up to the order of the Jordan blocks along the diagonal).

(c) Let $M_n(\mathbb{C})$ denote the set of $n \times n$ matrices with entries in $\mathbb{C}$. The group of invertible matrices $\mathrm{GL}_n(\mathbb{C})$ acts on $M_n(\mathbb{C})$ by conjugation. Conclude that in each conjugacy class there is a unique matrix in Jordan canonical form (up to the order of the blocks).

5. **(Building toward a theory of Jordan Canonical Form: Part 4 – Generalized Eigenspaces).** Let $T : V \to V$ be a linear map on a n-dimensional $\mathbb{C}$–vector space V. Let I denote the identity matrix. Recall that an *eigenvector* v of T with *eigenvalue* λ is defined to be a nonzero element of $\ker(\lambda I - T)$, and that the *eigenspace* E_λ is defined to be the subspace of V

$$E_\lambda = \ker(\lambda I - T) = \{\text{eigenvectors of } T \text{ with eigenvalue } \lambda\} \cup \{0\}$$

For an eigenvalue λ of T, define the *algebraic multiplicity* of λ to be the multiplicity of the root $(x-\lambda)$ in the characteristic polynomial of T, and the *geometric multiplicity* to be the $\dim_{\mathbb{C}}(E_\lambda)$. For an eigenvalue λ of T, define the *generalized eigenspace of* λ to be the subspace

$$G_\lambda = \{v \mid (\lambda I - T)^k v = 0 \text{ for some integer } k > 0\} \subseteq V$$

(a) Show (in a sentence) that $E_\lambda \subseteq G_\lambda$.

(b) Show that the geometric multiplicity of λ is equal to the number of Jordan blocks with diagonal entry λ in the Jordan canonical form of T.

(c) Show that the generalized eigenspace G_λ of V is precisely the direct sum of submodules of the form $\mathbb{C}[x]/(x-\lambda)^k$ in the decomposition of V.

(d) Conclude that V decomposes into a direct sum of generalized eigenspaces for T, and that the algebraic multiplicity of an eigenvalue λ is equal to sum of the sizes of the corresponding Jordan blocks, which is equal to the dimension of G_λ.

(e) Note as a corollary that geometric multiplicity of λ is no greater than the algebraic multiplicity of λ. Under what conditions are they equal?

(f) Briefly explain how you can compute the Jordan canonical form of a linear map T acting on V (which is uniquely defined up to order of the blocks) by computing its eigenvalues λ, and computing the dimensions of the $\ker(T-\lambda I)^m$ for each eigenvalue λ and $m \leq \dim_{\mathbb{C}}(V)$. **No justification needed.**

(g) State instructions for how to read off the following data from the Jordan canonical form of a linear map T, and state each for the specific map T_0 given below.
You do not need justify instructions or show your computations.

$$T_0 = \begin{bmatrix} 2 & 1 & & & & & & & & \\ & 2 & & & & & & & & \\ & & 2 & 1 & & & & & & \\ & & & 2 & & & & & & \\ & & & & 2 & 1 & & & & \\ & & & & & 2 & & & & \\ & & & & & & 2 & & & \\ & & & & & & & 2 & & \\ & & & & & & & & 3 & 1 \\ & & & & & & & & & 3 \end{bmatrix}$$

(i) The eigenvalues of T (with algebraic multiplicity).
(ii) The determinant of T.
(iii) The characteristic polynomial of T.
(iv) The minimal polynomial of T.
(v) The eigenvalues of T (with geometric multiplicity).

6. **(Bonus).** Let $\mathbb{F}$ be any field, and let A and B be $\mathbb{F}$–linear endomorphisms of $\mathbb{F}$–vector spaces V and W, respectively. Show that the trace of $A \otimes B$ on $V \otimes_{\mathbb{F}} W$ is the product $\mathrm{Trace}(A)\mathrm{Trace}(B)$.
You can use without proof the observation that the theory of Jordan Canonical Form you've developed applies equally well to $\overline{\mathbb{F}}[x]$–modules, where $\overline{\mathbb{F}}$ is the algebraic closure of $\mathbb{F}$. See Question 3(b).