Reading: Fulton–Harris Ch 2.1–3.4.

Summary of definitions and main results

Definitions we've covered: class function, character, character table, the inner product $\langle -, - \rangle_G$, induced representations, generalized eigenspaces.

Main results: irreducible characters form a basis for the space of class functions, orthogonality relations, Frobenius reciprocity.

Warm-Up Questions

1. Let G be a finite group and U an irreducible G–representation over $\mathbb{C}$.
   (a) Show that the G–representation $V \cong U \oplus U$ has infinitely many distinct direct sum decompositions into two copies of U.
   (b) Describe the $\mathbb{C}$–vector space of G–equivariant maps $\text{Hom}_{\mathbb{C}[G]}(U^{\oplus a}, U^{\oplus b})$.
   (c) Which of the maps $\text{Hom}_{\mathbb{C}[G]}(U^{\oplus a}, U^{\oplus b})$ are isomorphisms?
2. Suppose that G is a group with N_G conjugacy classes, and H a group with N_H conjugacy classes. Verify that $G \times H$ has $N_G N_H$ conjugacy classes.
3. Let G be a finite group and $\phi : G \to GL(V)$ a G–representation over a field $\mathbb{F}$ with character $\chi_V : G \to \mathbb{F}$. Prove that if V is 1-dimensional, then $\chi_V = \phi$. Show by example that if V is at least 2 dimensional, χ_V may not be a group homomorphism.
4. Let G be a finite group. Verify that $\langle -, - \rangle_G$ satisfies that conjugate symmetry, linearity, and positive definiteness properties that define an inner product.
5. Let G be a finite group.
   (a) State the formula for the inner product on complex-valued class functions of G.
   (b) Let $U = \sum_i V_i^{\oplus a_i}$ and $W = \sum_j V_j^{\oplus b_j}$ for distinct irreducible representations V_i. Compute $\langle \chi_W, \chi_U \rangle_G$.
   (c) Explain why the following results about character theory hold.
      (i) Characters of irreducible representations are orthonormal.
      (ii) Characters of irreducible representations are linearly independent.
      (iii) The number of irreducible representations is at most the number of conjugacy classes of G.
      (iv) A G–representation V is irreducible if and only if $\langle \chi_V, \chi_V \rangle_G = 1$.
      (v) A representation V is determined up to isomorphism by its character.
6. Let G be a finite group. Prove that a complex-valued class function on G is a character if and only if it is a nonnegative integer linear combination of irreducible characters.
7. Let G be a finite group. Prove that the dimension of the space of class functions $G \to \mathbb{F}$ over $\mathbb{F}$ is equal to the number of conjugacy classes of G.
8. Let G be a finite group. We saw in class that, as a module over itself, $\mathbb{C}[G] \cong \bigoplus_i V_i^{\oplus \dim_{\mathbb{C}}(V_i)}$, where $\{V_i\}$ is a complete set of non-isomorphic irreducible representations of G. What is the multiplicity of the trivial representation in $\mathbb{C}[G]$? Find a basis for this subrepresentation.
9. Let G be a group, and V and U be irreducible complex representations of G.
   (a) Show by example that $U \otimes_{\mathbb{C}} V$ may or may not be an irreducible G–representation.
   (b) Prove that if U is 1-dimensional, then $U \otimes_{\mathbb{C}} V$ is an irreducible G–representation.

10. Let G be a finite group.
    (a) If $\phi : G \to \mathrm{GL}(V)$ is a G–representation, prove that $\phi(g) : V \to V$ is G–equivariant if and only if $\phi(g)$ is central in $\phi(G)$.
    (b) Let χ be an irreducible character of G. Prove that for every element g in the center of G, $\chi(g) = \xi\chi(1)$, where ξ is a root of unity in $\mathbb{C}$.
11. Recall the character table for the complex representations of the symmetric group S_3.

	(•)(•)(•)	(••)(•)	(• • •)
<u>Trv</u>	1	1	1
<u>Alt</u>	1	−1	1
<u>Std</u>	2	0	−1

    (a) Let $\mathbb{C}^3$ denote the canonical permutation representation of S_3. Compute the character of $\underline{\mathrm{Alt}} \otimes_{\mathbb{C}} \mathrm{Sym}^2\mathbb{C}^3$.
    (b) Use the character table to decompose $\underline{\mathrm{Alt}} \otimes_{\mathbb{C}} \mathrm{Sym}^2\mathbb{C}^3$ as a sum of irreducible representations (in the sense of finding the multiplicity of each irreducible representation in the decomposition).
    (c) Verify that the orthogonality relations hold for this character table.
12. Find two non-isomorphic S_3–representations that are the same dimension. Explain why dimension is an isomorphism invariant of G–representations, but is not sufficient to distinguish non-isomorphic representations.
13. For $n \geq 2$, let $\mathbb{C}^n$ be the canonical permutation representation of S_n.
    (a) Prove that $\langle \chi_{\mathbb{C}^n}, \chi_{\mathbb{C}^n} \rangle_{S_n} = 2$.
    (b) Use this result to conclude that the standard representation is irreducible for every $n \geq 2$.
14. (a) Compute the character table of the cyclic group $G = \mathbb{Z}/4\mathbb{Z}$,
    (b) Verify the orthogonality relations on the row and columns of the character table.
    (c) Compute the character of $\bigwedge^3 \mathbb{C}[G]$, and determine its decomposition into irreducible characters.
15. Let G be a finite group and C be its character table (of all irreducible characters).
    (a) Show that the "orthogonality of characters" result is equivalent to the statement that the matrix C satisfies the relation $\overline{C}DC^T = I$ for a certain diagonal matrix D. What is D?
    (b) Conclude from this equation that $C^T\overline{C} = D^{-1}$. Use this equation to derive the second orthogonality result for characters.
    (c) Explicitly verify the relations $\overline{C}DC^T = I$ and $C^T\overline{C} = D^{-1}$ for the character table for S_3.
16. Prove that the character table is an invertible matrix.
17. Let G be a finite group and H a subgroup. Let e be the identity element of G.
    (a) Show that $\mathrm{Ind}_H^G \mathbb{C}[H] \cong \mathbb{C}[G]$. Note the special case $\mathrm{Ind}_{\{e\}}^G \mathbb{C} \cong \mathbb{C}[G]$.
    (b) Consider the trivial action of H on $\mathbb{C}$. Show that $\mathrm{Ind}_H^G \mathbb{C}$ is the permutation representation of G on the set of cosets G/H.
18. Use Frobenius reciprocity to perform the following computations.
    (a) Let $C_3 = \{1, (123), (321)\} \subseteq S_3$, and let V be the irreducible trivial C_3–representation. Find the decomposition of the induced S_3–representation $\mathrm{Ind}_{C_3}^{S_3} V$ into irreducible representations.
    (b) Do the same for the irreducible C_3–representation where (123) acts by mulitplication by $e^{\frac{2\pi i}{3}}$.
    (c) Let $C_2 = \{1, (12)\} \subseteq S_3$. Decompose the S_3–representations induced from the trivial and the nontrivial irreducible representations of C_2.

19. **(Linear Algebra Review).** Let V be a finite dimensional vector space over $\mathbb{C}$. For $a \in \mathbb{C}$, write $\overline{a}$ for its complex conjugate. Recall that a *Hermitian inner product* on V is a function

$$\langle -, - \rangle : V \times V \to \mathbb{C}$$

satisfying the following properties:

(1) (Conjugate symmetry)

$$\langle x, y \rangle = \overline{\langle y, x \rangle} \qquad \forall x, y \in V$$

(2) (Linearity in the first coordinate)

$$\langle ax, y \rangle = a\langle x, y \rangle \qquad \text{and} \qquad \langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle \qquad \forall x, y, z \in V, a \in \mathbb{C}$$

(3) (Positive definiteness)

$$\langle x, x \rangle \geq 0 \qquad \text{and} \qquad \langle x, x \rangle = 0 \Rightarrow x = 0 \qquad \forall x \in V$$

Observe that (1) and (2) imply that the Hermitian inner product is *antilinear* in the second coordinate:

$$\langle x, ay \rangle = \overline{a}\langle x, y \rangle \qquad \text{and} \qquad \langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle \qquad \forall x, y, z \in V, a \in \mathbb{C}$$

Remark: Compared to the bilinear form $(-, -)$ we studied when we defined dual spaces, the Hermitian inner product $\langle -, - \rangle$ has the advantage that it is positive definite, but the disadvantage that it is not linear in the second argument. These two definitions coincide when we work over $\mathbb{R}$ (instead of $\mathbb{C}$).

(a) Suppose that there is set of vectors $e_1, e_2, \ldots, e_n$ in V that is *orthonormal* with respect to the inner product $\langle -, - \rangle$. This means

$$\langle e_i, e_j \rangle = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Prove these vectors are linearly independent, and therefore form a basis for the space they span. (NB: We can always use the *Gram-Schmidt algorithm* to construct an orthonormal basis for V.)

(b) Let $v = a_1e_1 + \cdots + a_ne_n$ and $w = b_1e_1 + \cdots + b_ne_n$ be elements of V. Compute $\langle v, w \rangle$. Show in particular that

$$\langle v, e_i \rangle = a_i \qquad \text{and} \qquad \langle v, v \rangle = |a_1|^2 + |a_2|^2 + \cdots + |a_n|^2.$$

(c) Show that the function

$$\begin{aligned} || - || : V &\longrightarrow \mathbb{R}_{\geq 0} \\ ||v|| &= \sqrt{\langle v, v \rangle} \end{aligned}$$

defines a norm on V, and hence the function

$$\begin{aligned} d : V \times V &\longrightarrow \mathbb{R}_{\geq 0} \\ d(v, w) &= ||v - w|| \end{aligned}$$

defines a metric on V.

(d) Suppose that $v = a_1e_1 + \cdots + a_ne_n$ for **nonnegative integer** coefficients a_i. Show that

$$\langle v, v \rangle = a_1^2 + a_2^2 + \cdots + a_n^2,$$

and conclude that $\langle v, v \rangle = 1$ if and only if $v = e_i$ for some i.

(e) Suppose you have a function $\langle -, - \rangle : V \times V \to \mathbb{C}$ which you know satisfies the conjugate-symmetry and linearity properties of an inner product. Show that, if V has an basis that is orthonormal with respect to the function, then it must be positive definite.

Assignment Questions

1. (a) Let R be a commutative ring, and let S be an R–algebra. Given S–modules U, V, W, show that there is an isomorphism of R–modules

$$\mathrm{Hom}_S(V, U \oplus W) \cong \mathrm{Hom}_S(V, U) \oplus \mathrm{Hom}_S(V, W)$$

   A similar result (you don't need to prove) is that there is an isomorphism of R–modules

$$\mathrm{Hom}_S(V \oplus U, W) \cong \mathrm{Hom}_S(V, W) \oplus \mathrm{Hom}_S(U, W)$$

   (b) Let $\{V_i\}$ be a finite set of irreducible G–representations over $\mathbb{C}$. Let

$$U = \bigoplus V_i^{\oplus a_i} \qquad \text{and} \qquad W = \bigoplus V_j^{\oplus b_j} \qquad \text{for } a_i, b_j \in \mathbb{Z}_{\geq 0}.$$

   Compute $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}[G]}(U, W)$.

2. Let G be a finite group. In this question we will describe the ring structure on the group ring $\mathbb{C}[G]$. Let $V_1, \ldots, V_k$ denote a complete list of non-isomorphic irreducible complex G–representations.
   (a) The action of G on a representation V is equivalent to the data of a map of rings $\mathbb{C}[G] \to \mathrm{End}_{\mathbb{C}}(V)$, so we obtain a map of rings $\mathbb{C}[G] \to \bigoplus_{i=1}^k \mathrm{End}_{\mathbb{C}}(V_i)$. Show that this map is injective.
   (b) Conclude (by a dimension count) that there is an isomorphism of rings $\mathbb{C}[G] \cong \bigoplus_{i=1}^k \mathrm{End}_{\mathbb{C}}(V_i)$

3. (a) Compute the character table for the symmetric group S_5 over $\mathbb{C}$. *Hint:* Fulton–Harris Chapter 3.1.
   (b) Let $\mathbb{C}^5$ denote the canonical permutation representation of S_5. Use the character table to find the decomposition of $\mathrm{Sym}^2\mathbb{C}^5$ into irreducible S_5–representations.

4. (**Induced representations**) Suppose $H \subseteq G$ are finite groups, and $\mathbb{F}$ is a field. Given a finite dimensional G–representation W, we can restrict the action of G to the action of $H \subset G$. The resulting H–representation is denoted $\mathrm{Res}^G_H W$. Observe that $\mathrm{Res}^G_H W \cong W$ as $\mathbb{F}$–vector spaces.

   Conversely, given a finite dimensional group representation V of H over $\mathbb{F}$ (viewed as a $\mathbb{F}[H]$–module), we can construct a representation of G by extension of scalars. Since $\mathbb{F}[H]$ is a subring of $\mathbb{F}[G]$, we may view $\mathbb{F}[G]$ as a right $\mathbb{F}[H]$–module. Define a $\mathbb{F}[G]$–module, called the *induced representation* $\mathrm{Ind}^G_H V$, by

$$\mathrm{Ind}^G_H V := \mathbb{F}[G] \otimes_{\mathbb{F}[H]} V.$$

   (a) Cite properties of the tensor product to show that

$$\mathrm{Ind}^G_H(U \oplus U') \cong \mathrm{Ind}^G_H U \oplus \mathrm{Ind}^G_H U' \qquad \text{and} \qquad \mathrm{Ind}^G_K(\mathrm{Ind}^K_H V) \cong \mathrm{Ind}^G_H V$$

   for any representations U, U' of H or subgroups $H \subseteq K \subseteq G$.
   (b) Let G/H be the set of left cosets of G in H, and let $\{\sigma_i\}$ be a set of representatives of each coset. Show that

$$\mathbb{F}[G] \cong \bigoplus_{\sigma_i \in G/H} \mathbb{F}[H]$$

   as right $\mathbb{F}[H]$–modules.
   (c) Conclude that, as an abelian groups, $\mathrm{Ind}^G_H V \cong \bigoplus_{\sigma_i \in G/H} V$.
   (d) To promote this isomorphism of abelian groups to an isomorphism of $\mathbb{F}[G]$–modules, we make the following definition. For each coset representative σ_i we let $\sigma_i V$ be an isomorphic copy of the $\mathbb{F}$–vector space V where we denote the element v by $\sigma_i v$. Prove that there is a $\mathbb{F}[G]$–linear isomorphism

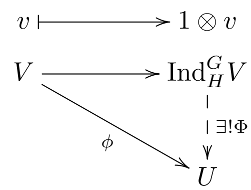
$$\mathrm{Ind}^G_H V \cong \bigoplus_{\sigma_i \in G/H} \sigma_i V$$

   where $g \in G$ acts on $\sigma_i v$ by finding the unique $h \in H$ and σ_j so that $g\sigma_i = \sigma_j h$, and then defining $g \cdot (\sigma_i v) = \sigma_j (h \cdot v)$.

(e) Given an G–representation W and H–representation V, find the degrees of $\operatorname{Res}_H^G W$ and $\operatorname{Ind}_H^G V$.

(f) What representation is $\operatorname{Ind}_H^G V$ when H is the trivial group and $V \cong \mathbb{F}$ the trivial representation?

(g) **(Universal property of induction)** Prove that induction satisfies the following universal property: If U is any representation of G, then any map of $\mathbb{F}[H]$–modules $\phi : V \to \operatorname{Res}_H^G U$ can be promoted uniquely to a map of $\mathbb{F}[G]$–modules $\Phi : \operatorname{Ind}_H^G V \to U$ making the following diagram commute.



(h) **(Ind-Res adjunction)** Show moreover that every $\mathbb{F}[G]$–module map $\operatorname{Ind}_H^G V \to U$ arises in this way. Conclude that there is a natural identification of $\mathbb{F}$–modules

$$\operatorname{Hom}_{\mathbb{F}[H]}(V, \operatorname{Res}_H^G U) \cong \operatorname{Hom}_{\mathbb{F}[G]}(\operatorname{Ind}_H^G V, U).$$

Remark: this is a special case of the tensor–Hom adjunction (Homework 7 Bonus Problem #5).

(i) **(Frobenius Reciprocity)** Conclude that for finite dimensional representations over $\mathbb{C}$,

$$\langle \chi_{\operatorname{Res}_H^G U}, \chi_V \rangle_H = \langle \chi_U, \chi_{\operatorname{Ind}_H^G V} \rangle_G.$$

(j) Conclude in particular that if V and U are irreducible representations of H and G, respectively, then the multiplicity of the $\mathbb{C}[H]$–representation V in $\operatorname{Res}_H^G U$ is equal to the multiplicity of the $\mathbb{C}[G]$–representation U in $\operatorname{Ind}_H^G V$.

5. **Bonus (optional).** Let V be an irreducible complex representation of a finite group G, and let H be an index-2 subgroup of G.

(a) Prove that $\operatorname{Res}_H^G V$ consists of either one or two irreducible H–representations. Prove moreover that the second case occurs if and only if $V \cong V \otimes_{\mathbb{C}} U$, where U is the 1-dimensional nontrivial representation $G \to G/H \cong \{\pm 1\} \subseteq GL(\mathbb{C})$.

(b) Suppose a group G has an abelian subgroup of index 2. Show that any irreducible representation of G has degree at most 2.

(c) Conclude that each irreducible complex representation of a dihedral group must have degree 1 or 2.

6. **Bonus (optional).** Compute the character tables for the dihedral groups D_5 (the symmetries of a pentagon) and D_6 (the symmetries of a hexagon).