Midterm Exam I

Math 122
3 May 2018
Jenny Wilson

Name: ______________________________

Instructions: This exam has 4 questions for a total of 20 points.

The exam is closed-book. No books, notes, cell phones, calculators, or other devices are permitted. Scratch paper is available.

Fully justify your answers unless otherwise instructed. You may cite any results from class or the homeworks without proof.

You have 2 hours to complete the exam. If you finish early, consider checking your work for accuracy.

Jenny is available to answer questions.

Question	Points	Score
1	4	
2	5	
3	5	
4	6	
Total:	20	

1. (a) (1 point) Define what it means for a ring R to be (left) Noetherian. You may use any of our equivalent definitions.

A ring R is *(left) Noetherian* if every R–submodule of R (viewed as left module over itself) is finitely generated.

(b) (3 points) Let R be the ring of continuous functions $f : \mathbb{R} \to \mathbb{R}$ under pointwise addition and pointwise multiplication (**not** composition). Show that R is **not** Noetherian.

Hint: For $n \in \mathbb{Z}_{\geq 0}$, consider $I_n = \{f \in R \mid f(x) = 0 \text{ for } x > n\} \subseteq R$.

By Homework #3, we can prove that R is not Noetherian by showing that the R–module R does not satisfy the ascending chain condition. Specifically, we will show that

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots$$

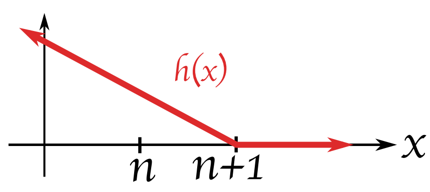
is an ascending chain of ideals that does not stabilize.

Fix n. First, we claim that I_n is an ideal: Given functions $f(x), g(x) \in I_n$, and $r(x) \in R$, we will check that $f(x) + r(x)g(x) \in I_n$. But for $x > n$,

$$f(x) + r(x)g(x) = 0 + r(x)0 = 0$$

as desired.

Next, to show that we have containment $I_n \subseteq I_{n+1}$, notice that if $f(x) \in I_n$, and $x > n + 1 > n$, then $f(x) = 0$, so $f(x) \in I_{n+1}$.
Finally, there exist continuous functions that are nonzero on $(-\infty, n + 1)$ and zero on $[n + 1, \infty)$ such as the piecewise-linear function $h(x)$ shown here:



The function $h(x)$ is contained in I_{n+1} but not I_n, so we have strict containment $I_n \subsetneq I_{n+1}$. Therefore

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots$$

is an ascending chain of ideals that does not stabilize.

2. Let R be an integral domain and M an R–module. Suppose that $x_1, \dots, x_n$ is a maximal (in cardinality) list of linearly independent elements in M. Let

$$N = Rx_1 + Rx_2 + \cdots + Rx_n.$$

(a) (3 points) Prove that M/N is a *torsion* R–module, that is, for every $m \in M/N$ there is some nonzero $r \in R$ so that $rm = 0$.

Let m be an element of M/N, and let $\overline{m}$ be any preimage of m in M under the quotient map $q : M \to M/N$.

Since the elements $x_1, \dots, x_n$ are assumed to be a maximal list of linearly independent elements, the list $\overline{m}, x_1, \dots, x_n$ is not linearly independent. Thus there are some coefficients $r, r_1, r_2, \dots, r_n \in R$, not all zero, so that

$$r\overline{m} + r_1x_1 + \cdots r_nx_n = 0.$$

Observe that the coefficient r must be nonzero, or this equation would violate the assumption that $x_1, x_2, \dots, x_n$ are linearly independent. We can write

$$r\overline{m} = -r_1x_1 - r_2x_2 - \cdots - r_nx_n,$$

which implies that $r\overline{m} \in N$. Hence

$$q(r\overline{m}) = rq(\overline{m}) = rm \qquad \text{is zero in the quotient } M/N,$$

and we conclude that m is a torsion element of M/N. Hence M/N is a torsion module as claimed.

(b) (2 points) State an example of an integral domain R, R–module M, and elements $x_1, \dots, x_n$ as above such that the quotient M/N is nonzero.
No justification needed.

Possible example. Let R=M=$\mathbb{Z}$. Since $\mathbb{Z}$ is cyclic, a maximal list of linearly independent elements has just one element. Let $x_1 = 2 \in \mathbb{Z}$. Then $\{x_1\}$ is linearly independent (since $2 \in \mathbb{Z}$ is not torsion) but $\mathbb{Z}/2\mathbb{Z}$ is nonzero.

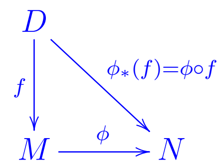
3. Let D, M, N be R–modules, and let $\phi : M \to N$ be an R–linear map. Consider the functor $\text{Hom}_R(D, -) : R\text{–}\underline{\text{Mod}} \to \underline{\text{Ab}}$.

(a) (1 point) State the definition of the induced map

$$\phi_* : \text{Hom}_R(D, M) \to \text{Hom}_R(D, N).$$

Given an R–linear map $f : D \to M$, the induced map ϕ_* is defined by

$$\phi_*(f) = \phi \circ f.$$



(b) (2 points) Show that if ϕ is injective, then so is the induced map ϕ_*.

Suppose that $f : D \to M$ is an element in the kernel of ϕ_*, and our goal is to show that f is the zero map. That f is in $\ker(\phi_*)$ means that

$$\phi_*(f) = \phi \circ f \qquad \text{must be the zero map in } \text{Hom}_R(D, N).$$

Hence, $f(D) \subseteq \ker(\phi)$. But ϕ is injective by assumption, so $f(D) = 0$. We conclude that f is the zero map, and so ϕ_* is injective, as claimed.

(c) (2 points) State an example of **nonzero** $\mathbb{Z}$–modules D, M, N and a $\mathbb{Z}$–linear map $\phi : M \to N$ such that the map ϕ is not the zero map, but the induced map ϕ_* is the zero map. **No justification needed.**

Possible example. Let $D = \mathbb{Z}/2\mathbb{Z}$, $M = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, and $N = \mathbb{Z}$. Define

$$\begin{aligned} \phi : \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} &\longrightarrow \mathbb{Z} \\ (a, b) &\longmapsto a \end{aligned}$$

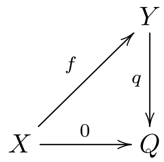
Any $\mathbb{Z}$–linear map $f : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ must have image contained in the 2-torsion subgroup $\mathbb{Z}/2\mathbb{Z} \subseteq \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, which is the kernel of ϕ. Hence $\phi_*(f) = \phi \circ f$ is always the zero map for every $f \in \text{Hom}_R(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})$.

Alternate example. Let $D = \mathbb{Z}/n\mathbb{Z}$, $M = \mathbb{Z}/mn\mathbb{Z}$, and $N = \mathbb{Z}/m\mathbb{Z}$ and the map

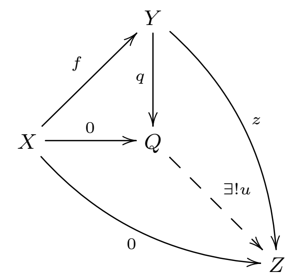
$$\begin{aligned} \phi : \mathbb{Z}/mn\mathbb{Z} &\longrightarrow \mathbb{Z}/m\mathbb{Z} \\ a &\longmapsto a \bmod m \end{aligned}$$

The map ϕ_* is zero since any map $D \to M$ must have image contained in the subgroup $\{0, m, 2m, 3m, \ldots, (n-1)m\} \in \mathbb{Z}/nm\mathbb{Z}$.

4. Let $\mathcal{C}$ be a category with a zero object $\mathbf{0}$, and therefore zero morphisms (denoted by 0). Let $f : X \to Y$ be a morphism. Define the *cokernel* of f to be an object Q along with a morphism $q : Y \to Q$ such that $q \circ f = 0$, that satisfies the following universal property.

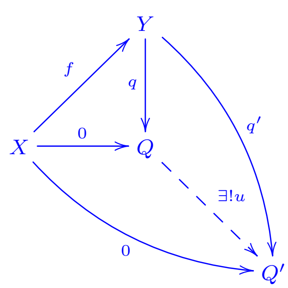


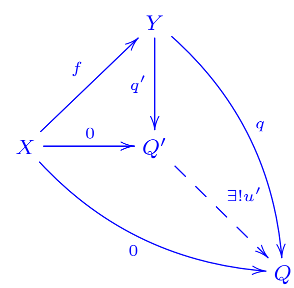
Given any object Z and $z : Y \to Z$ satisfying $z \circ f = 0$, there is a unique morphism u making the following diagram commute.



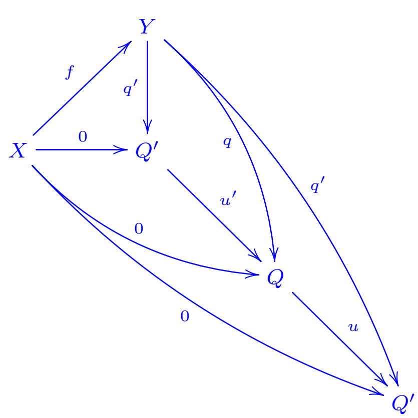
(a) (3 points) Prove that, if (Q, q) satisfying the universal property exist, then the universal property determines them uniquely up to unique isomorphism.

Suppose that (Q, q) and (Q', q') are both objects satisfying the universal property. Then applying the universal property for Q and Q', respectively, we obtain unique maps u and u', respectively, making the following diagrams commute. Our goal is to show these uniquely determined maps are isomorphisms.

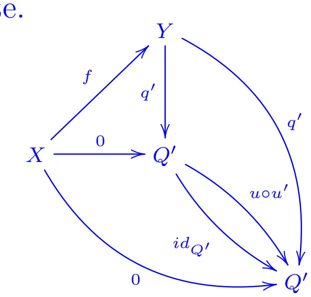




Next we apply the universal property to the outer triangle of the commutative diagram:



Both the map $id_{Q'} : Q' \to Q'$ and the map $u \circ u' : Q' \to Q'$ make the diagram commute.

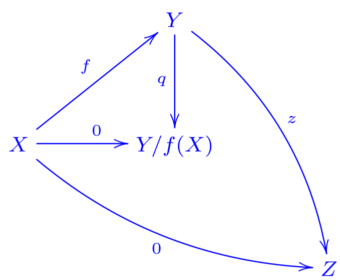


The universal property states that there is a unique map completing the diagram, so we conclude $u \circ u' = id_{Q'}$. Reversing the roles of Q' and Q, the same argument implies that $u' \circ u = id_Q$. Thus u and u' are isomorphisms, as claimed.

(b) (3 points) Let $\mathcal{C}$ be the category of R–modules. Show that the object $Y/f(X)$ (along with the quotient map $q : Y \to Y/f(X)$) satisfies the universal property, and is therefore the cokernel of f. You may quote the Factor Theorem without proof.

First observe that, for any $x \in X$, $(q \circ f)(x) = q(f(x)) = 0$ by construction, so $q \circ f = 0$ as required.

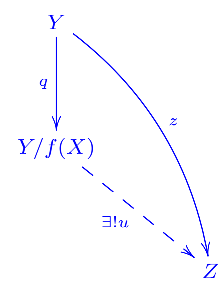
Now suppose we have a commutative diagram



Since $z \circ f = 0$, it follows that $f(X)$ is contained in $\ker(z)$. Hence, by the Factor Theorem, the map z factors uniquely through the quotient $Y/f(X)$. Specifically, the map

$$\begin{aligned} u : Y/f(X) &\longrightarrow Z \\ a + f(X) &\longmapsto z(a) \end{aligned}$$

is well-defined and is the unique map satisfying $u \circ q = z$.



Since $u \circ 0 = 0$, this map completes the diagram. Thus we have found the unique map u necessary to demonstrate that $Y/f(X)$ satisfies the universal property.