Midterm Exam II

Math 122
29 May 2018
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Name: ______________________________

Instructions: This exam has 4 questions for a total of 20 points.

The exam is closed-book. No books, notes, cell phones, calculators, or other devices are permitted. Scratch paper is available.

Fully justify your answers unless otherwise instructed. You may cite any results proved in class or on the homeworks without proof.

You have 2 hours to complete the exam. If you finish early, consider checking your work for accuracy.

Jenny is available to answer questions.

Question	Points	Score
1	4	
2	8	
3	2	
4	6	
Total:	20	

1. (a) (2 points) Suppose that V is a finite-dimensional $\mathbb{C}$–vector space, and $T : V \to V$ is a linear map with
   - minimal polynomial $(x-1)(x-3)^2$,
   - characteristic polynomial $(x-1)(x-3)^5$.

   Write down all the possibilities for the Jordan canonical form of T (up to the order of the Jordan blocks).
   No justification necessary. Blank entries in the matrices are understood to be 0.

   The map T has eigenvalues 1 and 3. The minimal polynomial implies that the largest Jordan block for $\lambda = 1$ is size 1×1, and the largest Jordan block for $\lambda = 3$ is size 2×2. The characteristic polynomial implies that the matrix must be 6×6, with 1 appearing once on the diagonal and 3 appearing five times. Therefore (up to the order of the Jordan blocks) there are 2 possibilities for the Jordan canonical form of T,

$$\begin{bmatrix} 1 & & & & & \\ & 3 & & & & \\ & & 3 & 1 & & \\ & & & 3 & & \\ & & & & 3 & 1 \\ & & & & & 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & & & & & \\ & 3 & & & & \\ & & 3 & & & \\ & & & 3 & & \\ & & & & 3 & 1 \\ & & & & & 3 \end{bmatrix}$$

   (b) (2 points) Let $\mathbb{C}^4$ denote the usual permutation representation of the symmetric group S_4. Compute the character of the induced action of S_4 on $\wedge^3\mathbb{C}^4$, and indicate its values on each conjugacy class of S_4 by completing the table below. **No justification necessary.**

	$(\bullet)(\bullet)(\bullet)(\bullet)$	$(\bullet\bullet)(\bullet\bullet)$	$(\bullet\bullet)(\bullet)(\bullet)$	$(\bullet\bullet\bullet)(\bullet)$	$(\bullet\bullet\bullet\bullet)$
$\wedge^3\mathbb{C}^4$	4	0	-2	1	0

Let e_1, e_2, e_3, e_4 denote the standard $\mathbb{C}$–basis for $\mathbb{C}^4$, so has $\mathbb{C}$–basis

$$\{e_1 \wedge e_2 \wedge e_3, \quad e_1 \wedge e_2 \wedge e_4, \quad e_1 \wedge e_3 \wedge e_4, \quad e_2 \wedge e_3 \wedge e_4\}.$$

We can compute the trace of a permutation σ by finding, for each of the four basis elements $e_i \wedge e_j \wedge e_k$, the component of $\sigma \cdot e_i \wedge e_j \wedge e_k$ in the direction of $e_i \wedge e_j \wedge e_k$, then summing these four coefficients. For example,

$$(12) \cdot e_1 \wedge e_2 \wedge e_3 = -e_1 \wedge e_2 \wedge e_3, \qquad (12) \cdot e_1 \wedge e_2 \wedge e_4 = -e_1 \wedge e_2 \wedge e_4$$
$$(12) \cdot e_1 \wedge e_3 \wedge e_4 = e_2 \wedge e_3 \wedge e_4, \qquad (12) \cdot e_2 \wedge e_3 \wedge e_4 = e_1 \wedge e_3 \wedge e_4$$

so $\text{Trace}((12)) = (-1) + (-1) + 0 + 0 = -2$.

2. Let V and W be vector spaces over a field $\mathbb{F}$.

   (a) (1 point) Let $\alpha : V \to \mathbb{F}$ and $\beta : W \to \mathbb{F}$ be $\mathbb{F}$–linear maps. Then α and β induce a map $\alpha \otimes \beta$ from $V \otimes_{\mathbb{F}} W$ to $\mathbb{F} \otimes_{\mathbb{F}} \mathbb{F} \cong \mathbb{F}$. Describe the map $\alpha \otimes \beta$ by indicating where in $\mathbb{F}$ it maps a simple tensor $v \otimes w \in V \otimes_{\mathbb{F}} W$.

$$\alpha \otimes \beta : V \otimes_{\mathbb{F}} W \longrightarrow \mathbb{F}$$

$$v \otimes w \longmapsto \underline{\qquad \alpha(v)\beta(w) \qquad}$$

   (b) (3 points) Show that this map $\alpha \otimes \beta$ is well-defined on $V \otimes_{\mathbb{F}} W$ and is $\mathbb{F}$–linear.

To show that the map $\alpha \otimes \beta$ is well-defined and $\mathbb{F}$–linear, by Homework 6 Question 2, it is enough to verify that it is induced by an $\mathbb{F}$–bilinear map

$$\begin{aligned} \phi : V \times W &\longrightarrow \mathbb{F} \\ (v, w) &\longmapsto \alpha(v)\beta(w) \end{aligned}$$

To check that this is $\mathbb{F}$–bilinear, let $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$, and $c_1, c_2 \in \mathbb{F}$. Then, using the $\mathbb{F}$–linearity of α and β, we find

$$\begin{aligned} &\phi(c_1 v_1 + c_2 v_2, w) \\ &= \alpha(c_1 v_1 + c_2 v_2)\beta(w) \\ &= \Big(c_1 \alpha(v_1) + c_2 \alpha(v_2)\Big)\beta(w) \\ &= c_1 \alpha(v_1)\beta(w) + c_2 \alpha(v_2)\beta(w) \\ &= c_1 \phi(v_1, w) + c_2 \phi(v_2, w) \end{aligned}$$

$$\begin{aligned} &\phi(v, c_1 w_1 + c_2 w_2) \\ &= \alpha(v)\beta(c_1 w_1 + c_2 w_2) \\ &= \alpha(v)\Big(c_1 \beta(w_1) + c_2 \beta(w_2)\Big) \\ &= c_1 \alpha(v)\beta(w_1) + c_2 \alpha(v)\beta(w_2) \\ &= c_1 \phi(v, w_1) + c_2 \phi(v, w_1) \end{aligned}$$

Thus the map ϕ is bilinear, and so we conclude that it factors through a (well-defined) and $\mathbb{F}$–linear map on $V \otimes_{\mathbb{F}} W$

$$\begin{aligned} V \otimes_{\mathbb{F}} W &\longrightarrow \mathbb{F} \\ v \otimes w &\longmapsto \alpha(v)\beta(w) \end{aligned}$$

as claimed.

(c) (4 points) Now suppose that V and W are finite dimensional vector spaces with bases $e_1, \ldots, e_n$ and $d_1, \ldots, d_m$, respectively. For clarity, for $\alpha \in V^*$ and $\beta \in W^*$, let's write $\alpha \otimes \beta$ to denote the element of the vector space $(V^* \otimes W^*)$, and write $\overline{\alpha \otimes \beta}$ to denote the linear map induced on $V \otimes W$ defined in part (a).
Show that there is a (well-defined) isomorphism of $\mathbb{F}$ vector spaces

$$\begin{aligned} \Phi : (V^* \otimes_{\mathbb{F}} W^*) &\xrightarrow{\cong} (V \otimes_{\mathbb{F}} W)^* \\ \alpha \otimes \beta &\longmapsto \overline{\alpha \otimes \beta} \end{aligned}$$

We first verify that Φ is a well-defined map of $\mathbb{F}$–vector spaces. By the universal property of the tensor product it suffices to verify that the map

$$\begin{aligned} \varphi : (V^* \times W^*) &\xrightarrow{\cong} (V \otimes_{\mathbb{F}} W)^* \\ (\alpha, \beta) &\longmapsto \overline{\alpha \otimes \beta} \end{aligned}$$

is $\mathbb{F}$–bilinear. So let $\alpha_1, \alpha_2 \in V^*$, $c_1, c_2 \in \mathbb{F}$, $\beta \in W^*$, $v \in V$, and $w \in W$, and consider the map $\varphi(c_1\alpha_1 + c_2\alpha_2, \beta) = \overline{((c_1\alpha_1 + c_2\alpha_2) \otimes \beta)}$.

$$\begin{aligned} \overline{((c_1\alpha_1 + c_2\alpha_2) \otimes \beta)}(v \otimes w) &= \Big((c_1\alpha_1 + c_2\alpha_2)(v)\Big)\Big(\beta(w)\Big) \\ &= c_1\alpha_1(v)\beta(w) + c_2\alpha_2(v)\beta(w) \\ &= c_1\overline{(\alpha_1 \otimes \beta)}(v \otimes w) + c_2\overline{(\alpha_2 \otimes \beta)}(v \otimes w) \\ &= \Big(c_1\overline{(\alpha_1 \otimes \beta)} + c_2\overline{(\alpha_2 \otimes \beta)}\Big)(v \otimes w) \end{aligned}$$

Since $\overline{((c_1\alpha_1 + c_2\alpha_2) \otimes \beta)}$ and $\Big(c_1\overline{(\alpha_1 \otimes \beta)} + c_2\overline{(\alpha_2 \otimes \beta)}\Big)$ take the same values on the generating set for $V \otimes_{\mathbb{F}} W$ of simple tensors, these maps must be equal, and we conclude that φ is $\mathbb{F}$–linear in the first coordinate. A similar argument shows that φ is $\mathbb{F}$–linear in the second coordinate, which allows us to conclude that the map is bilinear, as claimed.
By Homework 2 Question 5(c), V^* and W^* have bases $e_1^*, \ldots, e_n^*$ and $d_1^*, \ldots, d_m^*$, respectively, where

$$e_i^*(e_j) = d_i^*(d_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Then by Homework 6 Question 3(c), the vector space $(V \otimes_{\mathbb{F}} W)$ has $\mathbb{F}$–basis $\{e_i \otimes d_j\}$, and $(V^* \otimes_{\mathbb{F}} W^*)$ has $\mathbb{F}$–basis $\{e_i^* \otimes d_j^*\}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$.
The image of the basis element $e_i \otimes d_j$ under Φ is the map

$$\overline{e_i^* \otimes d_j^*} : (e_k \otimes d_\ell) \longmapsto e_i^*(e_k)d_j^*(d_\ell) = \begin{cases} 1, & i = k \text{ and } j = \ell \\ 0, & \text{otherwise} \end{cases}$$

which is precisely the dual basis element to the basis element $e_i \otimes d_j$ of $V \otimes_{\mathbb{F}} W$. Thus our map restricts to a bijection between the nm basis elements $\{e_i^* \otimes d_j^*\}$ for $V^* \otimes_{\mathbb{F}} W^*$, and the nm basis elements $\{(e_i \otimes d_j)^*\}$ for $(V \otimes_{\mathbb{F}} W)^*$. We conclude that Φ is an isomorphism, as claimed.

3. (2 points) Let R be a ring. If M is a cyclic R–module, must M be simple? Either give a proof, or state a counterexample (with justification).

   The statement is false. Consider the ring $R = \mathbb{Z}$, and the $\mathbb{Z}$–module $M = \mathbb{Z}/15\mathbb{Z}$. Then M is cyclic, since it is generated by the congruence class 1 mod 15. The module M is not simple, however, since it contains the proper nontrivial submodule $\{0, 5, 10\} \cong \mathbb{Z}/3\mathbb{Z}$. (The module M is, in fact, decomposable as the direct sum $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$).

4. Consider the action of S_2 on the abelian group $\mathbb{Z}^2$ by permuting the standard basis $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

   (a) (2 points) Now consider the trivial action of S_2 on $\mathbb{Z}$. Show that the map of $\mathbb{Z}$–modules

   $$\begin{aligned} \mathbb{Z}^2 &\xrightarrow{\varphi} \mathbb{Z} \\ (a, b) &\longmapsto a + b \end{aligned}$$

   is S_2–equivariant.

   We must check that φ commutes with the action of $S_2 = \{(12), (1)(2)\}$. Observe that

   $$\varphi\Big((1)(2) \cdot (a, b)\Big) = \varphi(a, b) = a + b = (1)(2) \cdot (a + b) = (1)(2) \cdot \varphi(a, b)$$

   $$\varphi\Big((12) \cdot (a, b)\Big) = \varphi(b, a) = b + a = a + b = (12) \cdot (a + b) = (12) \cdot \varphi(a, b)$$

   and so the map is S_2–equivariant as claimed.

(b) (4 points) The map φ extends to a short exact sequence of $\mathbb{Z}[S_2]$–modules

$$0 \longrightarrow \ker(\varphi) \longrightarrow \mathbb{Z}^2 \xrightarrow{\varphi} \mathbb{Z} \longrightarrow 0$$
$$(a,b) \longmapsto a+b$$

Either prove that this sequence of $\mathbb{Z}[S_2]$–modules splits, or prove that this sequence does not split.

We will prove that the sequence does not split. By the Splitting Lemma, the sequence splits if and only if there is a *splitting map*, a $\mathbb{Z}[S_2]$–linear map $\varphi' : \mathbb{Z} \to \mathbb{Z}^2$ such that $\varphi \circ \varphi' = \mathrm{id}_{\mathbb{Z}}$.

Suppose (for the sake of contradiction) that such a map φ' exists, and let (a,b) denote the element $\varphi'(1) \in \mathbb{Z}^2$. Because φ' is S_2–equivariant, and $(12)\cdot 1 = 1$, we find

$$(a,b) = \varphi(1) = \varphi\Big((12)\cdot 1\Big) = (12)\cdot\varphi(1) = (12)\cdot(a,b) = (b,a),$$

so $a = b$ and $\varphi'(1) = (a,a)$ for some $a \in \mathbb{Z}$.

However, since $\varphi \circ \varphi' = \mathrm{id}_{\mathbb{Z}}$, we deduce

$$1 = \varphi\Big(\varphi'(1)\Big) = \varphi(a,a) = a + a = 2a.$$

But the equation $1 = 2a$ has no solutions $a \in \mathbb{Z}$, so we have reached a contradiction, and we conclude that the sequence does not split.

Alternate solution outline. We can alternatively check that the $\ker(\varphi)$ does not have a direct complement in $\mathbb{Z}^2$. By direct computation, this kernel is the span of the vector $(1,-1)$ in $\mathbb{Z}^2$. A direct complement of this kernel must be a rank-1 subgroup of $\mathbb{Z}^2$. We can explicitly compute all the rank 1 S_2–invariant subgroups by determining those elements $(a,b) \in \mathbb{Z}^2$ such that $(12)\cdot(a,b) \in \mathrm{span}(a,b)$, that is,

$$(b,a) = c(a,b) \qquad \text{for } a,b,c \in \mathbb{Z}^2.$$

The possible solutions are elements of the form (a,a) or $(a,-a)$. So the only candidate direct complement to $\ker(\varphi)$ is the subgroup $\mathrm{span}(1,1)$. However, it is not the case that $\mathbb{Z}^2$ is the direct sum of $\ker(\varphi)$ and $\mathrm{span}(1,1)$ since their sum is a proper subgroup of $\mathbb{Z}^2$, for example, there are no integer solutions a,b to the equation

$$(a,a) + (b,-b) = (1,0).$$