## Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Give an intuitive geometric explanation of each of the 3 properties that define a metric.

2. Let $X = \{a, b, c\}$. Which of the following functions define a metric on $X$?

   (a)

$$\begin{aligned} d(a,a) = d(b,b) = d(c,c) &= 0 \\ d(a,b) = d(b,a) &= 1 \\ d(a,c) = d(c,a) &= 2 \\ d(b,c) = d(c,b) &= 3 \end{aligned}$$

   (b)

$$\begin{aligned} d(a,a) = d(b,b) = d(c,c) &= 0 \\ d(a,b) = d(b,a) &= 1 \\ d(a,c) = d(c,a) &= 2 \\ d(b,c) = d(c,b) &= 4 \end{aligned}$$

3. Consider the set $\mathbb{Z}$ with the Euclidean metric (defined by viewing $\mathbb{Z}$ as a subset of the metric space $\mathbb{R}$). What is the ball $B_3(1)$ as a subset of $\mathbb{Z}$? What is the ball $B_{\frac{1}{2}}(1)$?

4. Let $X = \mathbb{R}$ with the usual Euclidean metric $d(x, y) = |x - y|$.
   (a) Let $x$ and $r > 0$ be real numbers. Show that $B_r(x)$ is an open interval in $\mathbb{R}$. What are its endpoints?
   (b) Show that every interval of the real line the form $(a, b)$, $(-\infty, b)$, $(a, \infty)$, or $(-\infty, \infty)$ is open, for any $a < b \in \mathbb{R}$.
   (c) Show that the interval $[0, 1] \subseteq \mathbb{R}$ is closed.

5. Let $(X, d)$ be a metric space, and let $U \subseteq X$ be a subset. Does the set $U$ necessarily need to be either open or closed? Can it be neither? Can it be both?

6. The interval $[0, 1]$ is not open in the metric space $X = \mathbb{R}$ with the Euclidean metric, but it **is** open in the metric space $X = [0, 1]$ with the (restriction of the) Euclidean metric. Explain this surprising fact.

## Assignment questions

(Hand these questions in!)

1. Let $a < b \in \mathbb{R}$. Let $\mathcal{C}(a, b)$ denote the set of continuous functions from the closed interval $[a, b]$ to $\mathbb{R}$. Verify whether each of the following functions defines a metric on the set $\mathcal{C}(a, b)$.

(a)

$$\begin{aligned} d_1 : \mathcal{C}(a,b) \times \mathcal{C}(a,b) &\longrightarrow \mathbb{R} \\ d(f,g) &= \int_a^b |f(x) - g(x)| \, dx \end{aligned}$$

You may assume basic facts about integrals of continuous functions.

(b)

$$\begin{aligned} d_\infty : \mathcal{C}(a,b) \times \mathcal{C}(a,b) &\longrightarrow \mathbb{R} \\ d(f,g) &= \sup_{x \in [a,b]} |f(x) - g(x)| \end{aligned}$$

2. Is there a metric on the set $\mathbb{R}^2$ with the property that the set $\{(0,0)\}$ containing the single point $(0,0)$ is both open and closed? Find such a metric and prove that $\{(0,0)\}$ is both open and closed, or prove that no such metric exists.

3. (a) Prove *DeMorgan's Laws*: Let $X$ be a set and let $\{A_i\}_{i \in I}$ be a collection of subsets of $X$.

$$(i) \quad X \setminus \left( \bigcup_{i \in I} A_i \right) = \bigcap_{i \in I} (X \setminus A_i) \qquad\qquad (ii) \quad X \setminus \left( \bigcap_{i \in I} A_i \right) = \bigcup_{i \in I} (X \setminus A_i)$$

*Hint:* Remember that a good way to prove two sets $B$ and $C$ are equal to to prove that $B \subseteq C$ and that $C \subseteq B$!

(b) Let $(X,d)$ be a metric space, and let $\{C_i\}_{i \in I}$ be a collection of closed sets in $X$. Note that $I$ need not be finite, or countable! Prove that $\cap_{i \in I} C_i$ is a closed subset of $X$.

(c) Now let $(X,d)$ be a metric space, and let $\{C_i\}_{i=1}^n$ be a **finite** collection of closed sets in $X$. Prove that $\cup_{i \in I} C_i$ is a closed subset of $X$.

## Reflection

(Do this once for each time you present at the board in class, as part of your "participation" grade.)

On a separate piece of paper from your homework solutions, write

- your name;
- the day, worksheet number, and question number(s) you presented to the class;
- a brief reflection (eg, a few sentences) on something that you thought went well in your presentation, and/or something you want to do differently next time. These can be (for example) related to choice of math content, organization of the solution, clarity of the explanation, visual aids, boardwork, presentation delivery like speech and body language, or your responses to audience questions.

Your reflection will be graded just for completion, based on a good-faith effort to think critically about your presentation.