Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Let X, Y, Z be sets and let $f : X \to Y$ and $g : Y \to Z$ be functions.
   (a) Show that if f and g are both surjective, then $g \circ f$ is surjective.
   (b) Show that if f and g are both injective, then $g \circ f$ is injective.
   (c) Show by example that, if we only assume one of f and g is surjective, then $g \circ f$ need not be surjective.
   (d) Show by example that, if we only assume one of f and g is injective, then $g \circ f$ need not be injective.
2. Let $(X, \mathcal{T})$ be a topological space. Suppose that there is a path γ from a point $x \in X$ to a point $y \in X$. Use γ to write down the formula for a path from $y \in X$ to $x \in X$. *Hint:* How can you modify γ in order to traverse the path in the opposite direction?
3. Let $(X, \mathcal{T})$ be a topological space, and let $x \in X$. Show that there is a path from x to x.
4. Recall that we showed that $\mathbb{R}$ is path connected: given real numbers a and b, there is a "straight-line" path from a to b given by the function

$$\gamma : [0,1] \to \mathbb{R}$$
$$\gamma(t) = bt + a(t-1)$$

   Given points (a_1, a_2) and (b_1, b_2) in $\mathbb{R}^2$, write down the formula for a "straight-line" connecting these points. What about two points in $\mathbb{R}^n$?
5. Let $(X, \mathcal{T})$ be a topological space.
   (a) Let $(X, \mathcal{T})$ be a topological space. Explain why the condition tbhat X is compact is stronger than the assumption that X has a finite open cover.
   (b) Show that every topological space has a finite open cover. *Hint:* What is the first axiom of a topology?
6. (a) Give an example of a compact subset A of a topological space $(X, \mathcal{T})$, with a subset $B \subseteq A$ that is **not** compact.
   (b) Give an example of a noncompact subset A of a topological space $(X, \mathcal{T})$, with a subset $B \subseteq A$ that **is** compact.
7. Let $(X, \mathcal{T})$ be a topological space where points $\{x\}$ are closed sets. Explain why the condition that X is regular (Assignment Problem 5) is stronger than the condition that $(X, \mathcal{T})$ is Hausdorff.

Assignment questions

(Hand these questions in!)

1. Prove the following result.

Theorem (Generalized Intermediate Value Theorem). Let $(X, \mathcal{T}_X)$ be a connected topological space, and let $f : X \to \mathbb{R}$ be a continuous function (where the topology on $\mathbb{R}$ is induced by the Euclidean metric). If $x, y \in X$ and c lies between $f(x)$ and $f(y)$, then there exists $z \in X$ such that $f(z) = c$.

2. Prove that any continuous function $f : [0,1] \to [0,1]$ has a fixed point. (In other words, show that there is some $x \in [0,1]$ so that $f(x) = x$).
Hint: Consider the function

$$g : [0,1] \to \mathbb{R}$$
$$g(x) = f(x) - x.$$

3. Suppose that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are path-connected topological spaces. Show that the product $X \times Y$ with the product topology $\mathcal{T}_{X \times Y}$ is path-connected.

4. Suppose that $(X, \mathcal{T})$ is a Hausdorff topological space, and that C and D are compact subsets.
   (a) Show that $C \cup D$ is compact.
   (b) Show that $C \cap D$ is compact.

5. **Definition (Regular topological spaces).** A topological space $(X, \mathcal{T}_X)$ is called *regular* if, for every point $x \in X$ and every nonempty closed set C that does not contain x, there exist disjoint open sets U and V such that $x \in U$ and $C \subseteq V$.

   Suppose that $(X, \mathcal{T}_X)$ is a compact, Hausdorff topological space. Prove that X is regular.

6. (a) Let $[a, b]$ be a closed interval of $\mathbb{R}$ with the Euclidean metric. Prove that $[a, b]$ is compact.
   (b) Prove the following result.

   Theorem (Generalized Extreme Value Theorem). Let $(X, \mathcal{T}_X)$ be a compact topological space, and let $f : X \to \mathbb{R}$ be a continuous function (where the topology on $\mathbb{R}$ is induced by the Euclidean metric). Show there exists some $z \in X$ such that $f(z) = \sup(f(X))$. In other words, f achieves its supremum on X.

7. Prove that a compact subset of $\mathbb{R}^n$ (with the Euclidean metric) is sequentially compact.

Reflection

(Do this once for each time you present at the board in class, as part of your "participation" grade.)

On a separate piece of paper from your homework solutions, write

- your name;
- the day, worksheet number, and question number(s) you presented to the class;
- a brief reflection (eg, a few sentences) on something that you thought went well in your presentation, and/or something you want to do differently next time. These can be (for example) related to choice of math content, organization of the solution, clarity of the explanation, visual aids, boardwork, presentation delivery like speech and body language, or your responses to audience questions.

Your reflection will be graded just for completion, based on a good-faith effort to think critically about your presentation.