Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Let $(X, \mathcal{T})$ be a topological space, and $A \subseteq X$ a subset. Prove that the two following definitions of compactness are equivalent.
   - The subset A is *compact* if it is a compact topological space with respect to the subspace topology $\mathcal{T}_A$.
   - The subset A is *compact* if it satisfies the following property: for any collection of open subsets $\{U_i\}_{i \in I}$ of X such that $A \subseteq \bigcup_{i \in I} U_i$, there is a finite subscollection $U_1, U_2, \ldots, U_n$ such that $A \subseteq \bigcup_{i=1}^{n} U_i$.
2. Give an example of a subsets $A \subseteq B$ of $\mathbb{R}$ such that ...
   (a) A is compact, and B is noncompact
   (b) B is compact, and A is noncompact
3. Describe the connected components of $\mathbb{R}$ with the following topologies (Assignment Problem #1).
   (a) the topology induced by the Euclidean metric
   (b) the discrete topology
   (c) the indiscrete topology
   (d) the cofinite topology

Assignment questions

(Hand these questions in!)

0. **(Optional).** Submit your Math 490 course evaluation!
1. **Definition (Connected components of a topological space).** Let $(X, \mathcal{T}_X)$ be a topological space. A subset $C \subseteq X$ is called a *connected component* of X if
   (i) C is connected;
   (ii) if C is contained in a connected subset A, then $C = A$.

   (a) Show that any connected component of X is closed. (*Hint*: Homework 9, Problem 2).
   (b) Let $x \in X$. Show that the set
   $$\bigcup_{\substack{A \text{ is a connected set,} \\ x \in A}} A$$
   is a connected component of X.
   (c) Show that X is the **disjoint union** of its connected components. In other words, show that every point of X is contained in one, and only one, connected component.
   (d) Determine the connected components of $\mathbb{Q}$ (with the Euclidean metric). (Remember to rigorously justify your answer!)
   (e) Deduce from the example of $\mathbb{Q}$ that connected components need not be open.

(f) Suppose that X has the property that every point has a connected neighbourhood. Show that the connected components of X are open.

2. (a) Let (X, d) be a metric space. Suppose that $(a_n)_{n\in\mathbb{N}}$ is a sequence in X that contains no convergent subsequence. Prove that, for every $x \in X$, there is some $\epsilon_x > 0$ such that $B_{\epsilon_x}(x)$ contains only finitely many points of the sequence.

   (b) Prove that any compact metric space is sequentially compact.

3. (a) **Definition (Lebesgue number of an open cover).** Let (X, d) be a metric space, and let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of X. Then $\delta > 0$ is an *Lebesgue number*[1] for $\mathcal{U}$ if, for every $x \in X$, there is some associated index $i_x \in I$ such that $B_\delta(x) \subseteq U_{i_x}$.

   Suppose that (X, d) is a sequentially compact metric space. Prove that any open cover of X has a Lebesgue number $\delta > 0$.

   (b) **Definition (ϵ-nets of a metric space).** Let (X, d) be a metric space. A subset $A \subseteq X$ is called an ϵ-net if $\{B_\epsilon(a) \mid a \in A\}$ is an open cover of X.

   Suppose that (X, d) is a sequentially compact metric space, and $\epsilon > 0$. Prove that X has a finite ϵ-net.

   (c) Prove that any sequentially compact metric space is compact.

The last two exercises give the following theorem:

> **Theorem (Compactness vs sequential compactness).** Let (X, d) be a metric space. Then X is compact if and only if X is sequentially compact.

(This theorem does not hold, however, for abitrary topological spaces!)

Reflection

(Do this once for each time you present at the board in class, as part of your "participation" grade.)

On a separate piece of paper from your homework solutions, write

- your name;
- the day, worksheet number, and question number(s) you presented to the class;
- a brief reflection (eg, a few sentences) on something that you thought went well in your presentation, and/or something you want to do differently next time. These can be (for example) related to choice of math content, organization of the solution, clarity of the explanation, visual aids, boardwork, presentation delivery like speech and body language, or your responses to audience questions.

Your reflection will be graded just for completion, based on a good-faith effort to think critically about your presentation.

[1] Named for Henri Lebesgue, `https://en.wikipedia.org/wiki/Henri_Lebesgue`.