Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Rigorously prove that the following functions are continuous.

   (a) $f(x) = 5$ (b) $f(x) = 2x + 3$ (c) $f(x) = x^2$

   (d) $f(x) = g(x) + h(x)$, for continuous functions g and h.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = x^2 + 2$. Find the inverse images of the following sets, and verify that they are open.

   (a) $\mathbb{R}$ (b) $(-1, 1)$ (c) $(2, 3)$ (d) $(6, \infty)$

3. Let X and Y be metric spaces. Prove that the following functions are continuous.
   (a) Let $y_0 \in Y$, and let $f : X \to Y$ be the constant function given by $f(x) = y_0$ for all $x \in X$.
   (b) Let $g : X \to X$ be the identity function given by $g(x) = x$ for all $x \in X$.

4. See the definition of accumulation points and isolated points in Problem (1) below. Let $X = \mathbb{R}$. Find the set of accumulation points and the set of isolated points for each of the following subsets of X.

   (a) $S = \{0\}$ (b) $S = (0, 1)$ (c) $S = \mathbb{Q}$ (d) $S = \left\{ \frac{1}{n} \;\middle|\; n \in \mathbb{N} \right\}$

Assignment questions

(Hand these questions in!)

1. Consider the following definition.

   Definition (Accumulation points and isolated points of a set.) Let (X, d) be a metric space, and let $S \subseteq X$ be a set. A point $x \in X$ is called an *accumulation point* of S if every ball $B_r(x)$ around x contains at least one point of S distinct from x. (Note that x may or may not be an element of S). An element $s \in S$ that is **not** an accumulation point of S is called an *isolated point* of S.

   (a) Let (X, d) be a metric space and let $S \subseteq X$ be a **closed** subset. Let x be an accumulation point of S. Show that x is contained in S.
   (b) Let (X, d) be a metric space and let $S \subseteq X$ be any subset. Let x be an accumulation point of S, and let $B_r(x)$ be a ball centered around x of some radius $r > 0$. Show that $B_r(x)$ contains infinitely many points of S.

2. Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is called *open* if for every open set $U \subseteq X$, its image $f(U) \subseteq Y$ is open.
   (a) Give an example of metric spaces (X, d_X) and (Y, d_Y) and a function $f : X \to Y$ that is open, but not continuous.
   (b) Give an example of metric spaces (X, d_X) and (Y, d_Y) and a function $f : X \to Y$ that is continuous, but not open.

Remember to fully justify your solutions!

3. Prove the following theorem.

   Theorem (Equivalent Definition of Continuity.) Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be a function. Then f is continuous if and only if it satisfies the following property: for every closed set $C \subseteq Y$, the preimage $f^{-1}(C)$ is closed.

Reflection

(Do this once for each time you present at the board in class, as part of your "participation" grade.)

On a separate piece of paper from your homework solutions, write

- your name;
- the day, worksheet number, and question number(s) you presented to the class;
- a brief reflection (eg, a few sentences) on something that you thought went well in your presentation, and/or something you want to do differently next time. These can be (for example) related to choice of math content, organization of the solution, clarity of the explanation, visual aids, boardwork, presentation delivery like speech and body language, or your responses to audience questions.

Your reflection will be graded just for completion, based on a good-faith effort to think critically about your presentation.