Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Rigorously determine the limits of the following sequences of real numbers, or prove that they do not converge.

   (a) $a_n = 0$ (b) $a_n = \frac{1}{n^2}$ (c) $a_n = n$ (d) $a_n = (-1)^n$

2. Suppose that $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are sequences of real numbers that converge to a_∞ and b_∞, respectively. Prove that the sequence $(a_n + b_n)_{n\in\mathbb{N}}$ converges to $(a_\infty + b_\infty)$.

3. Consider the sequence $\left(\frac{(-1)^n}{n}\right)_{n\in\mathbb{N}}$ in $\mathbb{R}$. Let $\epsilon > 0$ be fixed. Find a number $N \in \mathbb{R}$ so that, for all $m, n \geq N$,
$$\left|\frac{(-1)^n}{n} - \frac{(-1)^m}{m}\right| < \epsilon.$$
   This shows that the sequence $\left(\frac{(-1)^n}{n}\right)_{n\in\mathbb{N}}$ is *Cauchy* (as defined in Question 2).

4. See the definition of bounded in Question 3.
   (a) Is $\varnothing$ a bounded set?
   (b) Show that any **finite** subset of a metric space is bounded.

5. Give examples of subsets of $\mathbb{R}$ (with the Euclidean metric) that satsify the following. See Question 3 for the definition of Bounded.

   (a) open, and bounded
   (b) closed, and bounded
   (c) open, and unbounded
   (d) closed, and unbounded

Assignment questions

(Hand these questions in!)

1. Let (X, d) be a metric space. Show that the limit of a sequence, if it exists, is *unique*, in the following sense. Suppose that $(a_n)_{n\in\mathbb{N}}$ is a sequence in X that converges to a point $a_\infty \in X$, and converges to a point $\widetilde{a}_\infty \in X$. Show that $a_\infty = \widetilde{a}_\infty$.

2. Consider the following definition.

   Definition (Cauchy sequence.) Let (X, d) be a metric space. Then a sequence $(a_n)_{n\in\mathbb{N}}$ of points in X is called a *Cauchy sequence* if for every $\epsilon > 0$ there exists some $N \in \mathbb{R}$ such that $d(a_n, a_m) < \epsilon$ whenever $n, m \geq N$.

   (a) Prove that every convergent sequence in X is a Cauchy sequence.
   (b) Give an example of a metric space (X, d) and a sequence $(a_n)_{n\in\mathbb{N}}$ in X that is Cauchy but does not converge. Remember to fully justify your solution!

3. Consider the following definition.

   Definition (Bounded subset.) Let (X, d) be a metric space. A subset $S \subseteq X$ is called *bounded* if there is some $x_0 \in X$ and some $R \in \mathbb{R}$ with $R > 0$ such that $S \subseteq B_R(x_0)$.

   (a) Let (X, d) be a metric space. Suppose that $(a_n)_{n\in\mathbb{N}}$ is a sequence in X converging to an element a_∞. Show that the set $\{ a_n \mid n \in \mathbb{N} \}$ is a bounded subset of X.

   (b) Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be a continuous function. Either prove the following statement, or construct (with proof) a counterexample: If $B \subseteq X$ is a bounded subset, then its image $f(B) \subseteq Y$ is bounded.

Reflection

(Do this once for each time you present at the board in class, as part of your "participation" grade.)

On a separate piece of paper from your homework solutions, write

- your name;
- the day, worksheet number, and question number(s) you presented to the class;
- a brief reflection (eg, a few sentences) on something that you thought went well in your presentation, and/or something you want to do differently next time. These can be (for example) related to choice of math content, organization of the solution, clarity of the explanation, visual aids, boardwork, presentation delivery like speech and body language, or your responses to audience questions.

Your reflection will be graded just for completion, based on a good-faith effort to think critically about your presentation.