Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. (a) Let (X, d) be a metric space, and $S \subseteq X$. Show that x is an accumulation point of S if and only if every neighbourhood of x contains a point of S distinct from x.

   (b) Let (X, d) be a metric space, and $S \subseteq X$. Show that x is in $\overline{S}$ if and only if every open ball $B_r(x)$ centred around x contains a point of S.

2. Consider the real numbers $\mathbb{R}$ with the Euclidean metric. For each of the following sets A, find $\mathring{A}$, $\overline{A}$, ∂A, $(\mathbb{R} \setminus A)^{\circ}$, $\overline{\mathbb{R} \setminus A}$, and $\partial(\mathbb{R} \setminus A)$. See Question 1 for the definition of a boundary ∂.

   (a) $\mathbb{R}$ (b) $[0, 1]$ (c) $(0, 1)$ (d) $\{0, 1\}$ (e) $\{\frac{1}{n} \mid n \in \mathbb{N}\}$

3. Consider the real numbers $\mathbb{R}$ with the Euclidean metric. Find examples of subsets A of $\mathbb{R}$ with the following properties.

   (a) $\partial(A) = \varnothing$

   (b) A has a nonempty boundary, and A contains its boundary ∂A.

   (c) A has a nonempty boundary, and A contains no points in its boundary

   (d) A has a nonempty boundary, and A contains some but not all of the points in its boundary.

   (e) A has a nonempty boundary, and $A = \partial A$.

   (f) A is a **proper** subset of ∂A.

4. Let X be a nonempty set with the discrete metric. Let $A \subseteq X$. Show that $A = \mathring{A} = \overline{A}$. Conclude that $\partial A = \varnothing$.

Assignment questions

(Hand these questions in!)

1. Consider the following definition.

   Definition (Boundary of a set A.) Let (X, d) be a metric space, and let $A \subseteq X$. Then the *boundary* of A, denoted ∂A, is the set $\overline{A} \setminus \mathring{A}$.

   Let (X, d) be a metric space, and let $A \subseteq X$.

   (a) Prove that $\mathring{A} = \overline{A} \setminus \partial A$.

   (b) Prove that $\partial A = \overline{A} \cap \overline{(X \setminus A)}$.

   (c) Conclude from part (b) that ∂A is closed.

   (d) Additionally conclude from part (b) that $\partial A = \partial(X \setminus A)$.

   (e) Prove the following characteriziation of points in the boundary:

   Theorem (An equivalent definition of ∂A). Let (X, d) be a metric space, and let $A \subseteq X$. Then $x \in \partial A$ if and only if every ball $B_r(x)$ about x contains at least one point of A, and at least one point of $X \setminus A$.

(f) Deduce that we can classify every point of X in one of three mutually exclusive categories:
   (i) interior points of A;
   (ii) interior points of $X \setminus A$;
   (iii) points in the (common) boundary of A and $X \setminus A$.

2. Prove the following equivalent definition of continuity.

   Theorem (An equivalent definition of continuity). Let (X, d_X) and (Y, d_Y) be metric spaces. Then a map $f : X \to Y$ is continuous if and only if

$$f\left(\overline{A}\right) \subseteq \overline{f(A)} \qquad \text{for every subset } A \subseteq X.$$

3. Consider the real numbers $\mathbb{R}$ with the Euclidean metric. Determine the interior, closure, and boundary of the subset $\mathbb{Q} \subseteq \mathbb{R}$. Remember to rigorously justify your solution!

Reflection

(Do this once for each time you present at the board in class, as part of your "participation" grade.)

On a separate piece of paper from your homework solutions, write

- your name;
- the day, worksheet number, and question number(s) you presented to the class;
- a brief reflection (eg, a few sentences) on something that you thought went well in your presentation, and/or something you want to do differently next time. These can be (for example) related to choice of math content, organization of the solution, clarity of the explanation, visual aids, boardwork, presentation delivery like speech and body language, or your responses to audience questions.

Your reflection will be graded just for completion, based on a good-faith effort to think critically about your presentation.