Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Let (X, d) be a metric space and $x \in X$. Suppose $r, R \in \mathbb{R}$ satisfy $0 < r < R$. Prove that $B_r(x) \subseteq B_R(x)$.

2. Let (X, d) be a metric space, and let $(a_n)_{n\in\mathbb{N}}$ be a sequence of points in X. Recall that we proved that, if $\lim_{n\to\infty} a_n = a_\infty$, then any subsequence of $(a_n)_{n\in\mathbb{N}}$ also converges to a_∞.
   (a) Suppose that $(a_n)_{n\in\mathbb{N}}$ has a subsequence that does not converge. Prove that $(a_n)_{n\in\mathbb{N}}$ does not converge.
   (b) Suppose that $(a_n)_{n\in\mathbb{N}}$ has a subsequence converging to $a \in X$, and a different subsequence converging to $b \in X$, with $a \neq b$. Prove that $(a_n)_{n\in\mathbb{N}}$ does not converge.

3. Let (X, d) be a metric space, and let $(a_n)_{n\in\mathbb{N}}$ be a sequence of points in X. Suppose that the set $\{a_n \mid n \in \mathbb{N}\}$ is unbounded. Explain why $(a_n)_{n\in\mathbb{N}}$ cannot converge.

4. Find examples of sequences $(a_n)_{n\in\mathbb{N}}$ of real numbers with the following properties.
   (a) $\{a_n \mid n \in \mathbb{N}\}$ is unbounded, but $(a_n)_{n\in\mathbb{N}}$ has a convergent subsequence
   (b) $(a_n)_{n\in\mathbb{N}}$ has no convergent subsequences
   (c) $(a_n)_{n\in\mathbb{N}}$ is not an increasing sequence, but it has an increasing subsequence
   (d) $(a_n)_{n\in\mathbb{N}}$ has four subsequences that each converge to a distinct limit point

5. Let X be a metric space with the discrete metric.
   (a) Suppose X is a finite set. Is X sequentially compact?
   (b) Suppose X is an infinite set. Is X sequentially compact?

Assignment questions

(Hand these questions in!)

1. Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be a continuous map. Prove that if $S \subseteq X$ is sequentially compact, then $f(S) \subseteq Y$ is sequentially compact.

2. Consider the following definition.

   Definition (Complete metric spaces.) A metric space (X, d) is called *complete* if every Cauchy sequence in X converges.

   Suppose that (X, d) is sequentially compact. Show that (X, d) is complete.

3. Recall the following result from real analysis (which you do not need to prove):

   Theorem (Bolzano–Weierstrass.) Let $S \subseteq \mathbb{R}^n$ be a bounded infinite set. Then S has an accumulation point $x \in \mathbb{R}^n$.

   Prove the following:

Theorem (Sequential compactness in $\mathbb{R}^n$.) Consider the space $\mathbb{R}^n$ with the Euclidean metric. Let $S \subseteq \mathbb{R}^n$ be a (finite or infinite) subset. Then S is sequentially compact if and only if S is closed and bounded.

4. Let (X, d_X) and (Y, d_Y) be metric spaces. Define

$$\begin{aligned} d_{X\times Y} :\ (X \times Y) \times (X \times Y) &\longrightarrow \mathbb{R} \\ d_{X\times Y}((x_1, y_1), (x_2, y_2)) &= \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2}. \end{aligned}$$

   (a) Prove that $X \times Y$ is a metric space with metric given by the function $d_{X\times Y}$.

   (b) Prove that if $U \subseteq X$ and $V \subseteq Y$ are open sets, then $U \times V$ is an open subset of $X \times Y$.

Reflection

(Do this once for each time you present at the board in class, as part of your "participation" grade.)

On a separate piece of paper from your homework solutions, write

- your name;
- the day, worksheet number, and question number(s) you presented to the class;
- a brief reflection (eg, a few sentences) on something that you thought went well in your presentation, and/or something you want to do differently next time. These can be (for example) related to choice of math content, organization of the solution, clarity of the explanation, visual aids, boardwork, presentation delivery like speech and body language, or your responses to audience questions.

Your reflection will be graded just for completion, based on a good-faith effort to think critically about your presentation.