Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Let (X, d) be a metric space and let $S \subseteq X$ be a bounded set. Show that any subset of S is bounded.
2. Let (X, d) be a metric space. Verify that the collection $\mathcal{T}_d$ of open sets in this metric space does indeed form a topology on the set X.
3. Find all possible topologies on the set $X = \{0, 1\}$.
4. Let $X = \{0, 1, 2\}$. Show that the collection of subsets $\Big\{\varnothing, X, \{0, 1\}, \{1, 2\}\Big\}$ is **not** a topology on X.
5. Let $(X, \mathcal{T})$ be a topological space.
   (a) Show by induction that the intersection of any **finite** collection of open sets is open.
   (b) Explain why this argument does not apply to an infinite collection of open sets.
6. Let X be a set. Show that the discrete topology on X is induced by the discrete metric on X.
7. Let X be a set. See the definition of the cofinite topology on X in Assignment Problem 3. Show that if X is a finite set, then the cofinite topology coincides with the discrete topology on X.
8. Give an example of a metric space (X, d) and a continuous function $f : X \to \mathbb{R}$ such that f has a finite supremum on X, but f does not achieve its supremum at any point $x \in X$.

Assignment questions

(Hand these questions in!)

1. Let (X, d) be a metric space and C a sequentially compact subset of X. Let $f : X \to \mathbb{R}$ be a continuous function. Prove that there is a point $c \in C$ so that

$$f(c) = \sup_{x \in C} f(x).$$

   In other words, prove that f achieves its supremum on C.

2. Let (X, d_X) and (Y, d_Y) be metric spaces. Recall from Homework 5 that $X \times Y$ is then a metric space with metric

$$\begin{aligned} d_{X\times Y} :\ & (X \times Y) \times (X \times Y) \longrightarrow \mathbb{R} \\ d_{X\times Y}\Big((x_1, y_1), (x_2, y_2)\Big) &= \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2}. \end{aligned}$$

   (a) Show that the projection map

$$\begin{aligned} \pi_X : (X \times Y, d_{X\times Y}) &\longrightarrow (X, d_X) \\ \pi_X(x, y) &= x \end{aligned}$$

is continuous. The same argument (which you do not need to repeat) shows that the projection map

$$\pi_Y : (X \times Y, d_{X\times Y}) \longrightarrow (Y, d_Y)$$
$$\pi_X(x, y) = y$$

is continuous.

(b) Let $U \subseteq X \times Y$ be an open set, and let $(x, y) \in U$. Show that there is a neighbourhood U_x of x in X and a neighbourhood U_y of y in Y so that $U_x \times U_y \subseteq U$.

(c) Let (Z, d_Z) be a metric space, and suppose that $f : Z \to X$ and $g : Z \to Y$ are continuous functions. Prove that the function

$$(f \times g) : Z \longrightarrow X \times Y$$
$$(f \times g)(z) = \Big(f(z), g(z)\Big)$$

is continuous.

3. Let X be a set, and let $\mathcal{T}$ be the collection of subsets

$$\mathcal{T} = \{\varnothing\} \cup \{U \subseteq X \mid X \setminus U \text{ is a finite set}\}.$$

Verify that $\mathcal{T}$ is a topology on X. It is called the *cofinite topology.*

Reflection

(Do this once for each time you present at the board in class, as part of your "participation" grade.)

On a separate piece of paper from your homework solutions, write

- your name;
- the day, worksheet number, and question number(s) you presented to the class;
- a brief reflection (eg, a few sentences) on something that you thought went well in your presentation, and/or something you want to do differently next time. These can be (for example) related to choice of math content, organization of the solution, clarity of the explanation, visual aids, boardwork, presentation delivery like speech and body language, or your responses to audience questions.

Your reflection will be graded just for completion, based on a good-faith effort to think critically about your presentation.