Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Prove the following theorem. *Hint:* See Homework #2 Problem 3.

   Theorem (Equivalent definition of continuity.) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Then a map $f : X \to Y$ is continuous if and only if for every closed set $C \subseteq Y$, the set $f^{-1}(C)$ is closed.

2. Prove the following theorem. *Hint:* See Worksheet #3 Problem 2.

   Theorem (Composition of continuous functions.) Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$, and $(Z, \mathcal{T}_Z)$ be topological spaces. Suppose that $f : X \to Y$ and $g : Y \to Z$ are continuous maps. Prove that the map $g \circ f : X \to Z$ is continuous.

3. Let $(X, \mathcal{T})$ be a topological space.
   (a) Show that $\mathcal{T}$ is a basis for $\mathcal{T}$.
   (b) Suppose that $\mathcal{B}$ is a basis for $\mathcal{T}$. Show that any collection of open sets in X containing $\mathcal{B}$ is also a basis for $\mathcal{T}$.

4. Verify that the set of open intervals $(a, b) \subseteq \mathbb{R}$ is a basis for $\mathbb{R}$ (with the Euclidean topology).

Assignment questions

(Hand these questions in!)

1. (a) Let $(X, \mathcal{T}_X)$ be a topological space, and let $S \subseteq X$ be a subset of X. Show that the collection of sets
   $$\mathcal{T}_S = \{\, U \cap S \mid U \in \mathcal{T}_X \,\}$$
   is a topology on S. The topology $\mathcal{T}_S$ is called the *subspace topology* on S.

   (b) Let $(X, \mathcal{T}_X)$ be a topological space and let $S \subseteq X$ be a subset endowed with the subset topology $\mathcal{T}_S$. Show that a set $C \subseteq S$ is closed if and only if there is some set $D \subseteq X$ that is closed with $C = D \cap S$.

2. Consider the following definition.

   Definition (Hausdorff space.) A topological space $(X, \mathcal{T})$ is called *Hausdorff*[1] if for any pair of distinct points x and y, there exist **disjoint** open sets $U \subseteq X$ and $V \subseteq X$ so with $x \in U$ and $y \in V$.

   (a) Let (X, d) be a metric space. Show that the topology induced by the metric d is Hausdorff.
   (b) Let X be an infinite set, and let $\mathcal{T}$ be the cofinite topology on X (defined in Homework #6, Problem 3). Show that $\mathcal{T}$ is not metrizable.

[1] Named for Felix Hausdorff, `https://en.wikipedia.org/wiki/Felix_Hausdorff`

3. Let X be a set and let $\mathcal{B}$ be a collection of subsets of X such that

   - $\displaystyle\bigcup_{B\in\mathcal{B}} B = X$
   - If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ then there is some $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq (B_1 \cap B_2)$.

   Let $\mathcal{T}$ be the collection of subsets of X

   $$\{\ U \mid U \text{ is a union of elements of } \mathcal{B}\ \}.$$

   Prove that $\mathcal{T}$ is a topology on X, and that $\mathcal{B}$ is a basis for $\mathcal{T}$.

4. (a) Consider the following definition.

   Definition (The product topology). Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Then the *product topology* $\mathcal{T}_{X\times Y}$ on $X \times Y$ is the collection of subsets of $X \times Y$ generated by the set

   $$\mathcal{B} = \{U \times V \mid\ U \subseteq X \text{ is open, and } V \subseteq Y \text{ is open}\}.$$

   (Recall that this means $\mathcal{T}_{X\times Y}$ consists of all unions of elements of $\mathcal{B}$.) Verify that $\mathcal{T}_{X\times Y}$ is indeed a topology on $X \times Y$, and that $\mathcal{B}$ is a basis for this topology.

   (b) Prove the following theorem.

   Theorem (Equivalent definition of the product topology). Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Then the product topology $\mathcal{T}_{X\times Y}$ on $X \times Y$ is precisely the collection of subsets of $X \times Y$,

   $$\mathcal{T}_{X\times Y} = \left\{ W \;\middle|\; \begin{array}{l} \text{for each } (x,y) \in W, \text{ there is a some } U \in \mathcal{T}_X \text{ and } V \in \mathcal{T}_Y \\ \text{such that } (x,y) \in (U \times V) \subseteq W \end{array} \right\}.$$

Reflection

(Do this once for each time you present at the board in class, as part of your "participation" grade.)

On a separate piece of paper from your homework solutions, write

- your name;
- the day, worksheet number, and question number(s) you presented to the class;
- a brief reflection (eg, a few sentences) on something that you thought went well in your presentation, and/or something you want to do differently next time. These can be (for example) related to choice of math content, organization of the solution, clarity of the explanation, visual aids, boardwork, presentation delivery like speech and body language, or your responses to audience questions.

Your reflection will be graded just for completion, based on a good-faith effort to think critically about your presentation.