## Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Let $(X, \mathcal{T})$ be a topological space. Let $S \subseteq X$ and let $\mathcal{T}_S$ be the subspace topology on $S$. Prove that if $S$ is an open subset of $X$, and if $U \in \mathcal{T}_S$, then $U \in \mathcal{T}$.

2. (a) Suppose that $(X, \mathcal{T})$ is a topological space with the property that the singleton set $\{x\}$ is open for ever $x \in X$. Prove that $\mathcal{T}$ is the discrete topology on $X$.

   (b) Suppose that $(X, \mathcal{T})$ is a topological space with the property that the singleton set $\{x\}$ is closed for ever $x \in X$. Must $\mathcal{T}$ be the discrete topology on $X$?

3. Let $X$ be a set, and $A \subseteq X$ a proper subset. What are the interior and closure (Assignment Problems #1 and #2) of $A$ if $X$ is given

   (a) the discrete topology?

   (b) the indiscrete topology?

4. Let $X = \{a, b, c, d\}$. Let $\mathcal{T}$ be the topology on $X$

$$\mathcal{T} = \{\varnothing, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}, X\}.$$

   Find the interior and closure of the subsets

   (a) $\{a, b, c\}$ (b) $\{a, c, d\}$ (c) $\{a, b, d\}$ (d) $\{b\}$ (e) $\{d\}$ (f) $\{b, d\}$

5. Let $X = \{a, b, c, d\}$. Let $\mathcal{T}$ be the topology on $X$

$$\mathcal{T} = \{\varnothing, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}, X\}.$$

   (a) Which elements of $X$ are limits of the constant sequence $x_n = d$? The constant sequence $x_n = a$? The constant sequence $x_n = b$?

   (b) Give an example of a sequence in $X$ that does not converge.

6. Let $X = \{0, 1\}$. Find a topology on $X$ for which the following sequence converges:

$$0\ 1\ 0\ 1\ 0\ 1\ 0\ 1 \cdots$$

7. Suppose that $(X, \mathcal{T})$ is a Hausdorff topological space, and let $S \subseteq X$. Verify that the subspace topology $\mathcal{T}_S$ on $S$ is Hausdorff.

8. Suppose that $(X, \mathcal{T})$ is a topological space, and that $(a_n)_{n \in \mathbb{N}}$ is a sequence in $X$ that converges to $a_\infty \in X$. Prove that any subsequence of $(a_n)_{n \in \mathbb{N}}$ converges to $a_\infty \in X$.

## Assignment questions

(Hand these questions in!)

1. (a) Let $(X, d)$ be a metric space, and let $A \subseteq X$. Prove that $a$ is an interior point of $A$ if and only if there is a neighbourhood $U$ of $a$ such that $U \subseteq A$.

(b) **Definition (Interior of a set in a topological space).** Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. Define the *interior* of $A$ to be the set

$$\mathring{A} = \{\, a \in A \mid \text{there is some neighbourhood } U \text{ of } a \text{ such that } U \subseteq A. \,\}$$

Prove that $\mathring{A}$ is necessarily an open set.

(c) Suppose that $A \subseteq X$ is any subset, and $U \subseteq A$ is an open set. Prove that $U \subseteq \mathring{A}$.

2. (a) **Definition (Closure of a set in a topological space).** Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. Define the *closure* of $A$ to be the set

$$\overline{A} = \{\, x \in X \mid \text{any neighbourhood } U \text{ of } x \text{ contains a point of } A \,\}.$$

Prove that $A \subseteq \overline{A}$.

(b) Prove that $\overline{A}$ is necessarily a closed set.

(c) Suppose that $A \subseteq X$ is any subset, and $C$ is a closed set containing $A$. Prove that $\overline{A} \subseteq C$.

3. Suppose that $(X, \mathcal{T}_X)$ is a topological space, and that $(Y, \mathcal{T}_Y)$ is a **Hausdorff** topological space. Let $f : X \to Y$ and $g : X \to Y$ be continuous functions. Suppose that $A \subseteq X$ is a subset such that

$$f(a) = g(a) \qquad \text{for all } a \in A.$$

Prove that

$$f(x) = g(x) \qquad \text{for all } x \in \overline{A}.$$

This says that the values of a continuous function on $\overline{A}$ are completely determined by its values on $A$.

## Reflection

(Do this once for each time you present at the board in class, as part of your "participation" grade.)

On a separate piece of paper from your homework solutions, write

- your name;
- the day, worksheet number, and question number(s) you presented to the class;
- a brief reflection (eg, a few sentences) on something that you thought went well in your presentation, and/or something you want to do differently next time. These can be (for example) related to choice of math content, organization of the solution, clarity of the explanation, visual aids, boardwork, presentation delivery like speech and body language, or your responses to audience questions.

Your reflection will be graded just for completion, based on a good-faith effort to think critically about your presentation.