Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Let $(X, \mathcal{T})$ be a topological space. Show that any subset $A = \{x\} \subseteq X$ of a single element is connected.

2. (a) Show that, for $a, b \in \mathbb{R}$, the subsets $\varnothing, \{a\}, (a, b), (a, b], [a, b), [a, b], (a, \infty), [a, \infty), (\infty, b), (\infty, b]$, and $\mathbb{R}$ of $\mathbb{R}$ are all intervals in the sense of Problem 3.
   (b) Show that every interval must have one of these forms.

3. Let $X = \{0, 1\}$.
   (a) Consider X as a topological space with the discrete topology. Rigorously show that X is not path-connected, and that X is not connected.
   (b) Consider X as a topological space with the indiscrete topology. Rigorously show that X is connected and path-connected.

4. Let $X = \{a, b, c, d\}$ with the topology
$$\mathcal{T} = \{\varnothing, \{a\}, \{a, b\}, \{c\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}.$$
   (a) Is X Hausdorff?
   (b) Is X connected?
   (c) What is the interior of $\{a, c, d\}$?
   (d) What is the closure of $\{a, c, d\}$?
   (e) Find a proper subset of X that is connected, and a proper subset of X that is disconnected.

Assignment questions

(Hand these questions in!)

1. Let $(X, \mathcal{T}_X)$ be a topological space, and let $A, B \subseteq X$.
   (a) Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
   (b) Show by example that $\overline{A \cap B}$ need not equal $\overline{A} \cap \overline{B}$.

2. Let $(X, \mathcal{T}_X)$ be a topological space, and let $A \subseteq X$ be a connected subset. Let B be any subset such that $A \subseteq B \subseteq \overline{A}$. Prove that B is connected.

   Remark: This shows in particular that if A is connected, then so is $\overline{A}$.

3. Recall the Intermediate Value Theorem from real analysis (which you may use without proof).

   Intermediate Value Theorem. If $f : [a, b] \to \mathbb{R}$ is continuous and d lies between $f(a)$ and $f(b)$ (i.e. either $f(a) \leq d \leq f(b)$ or $f(b) \leq d \leq f(a)$), then there exists $c \in [a, b]$ such that $f(c) = d$.

   Define a subset $A \subseteq \mathbb{R}$ to be an *interval* if whenever $x, y \in A$ and z lies between x and y, then $z \in A$.

(a) Prove that any interval of $\mathbb{R}$ is connected.

(b) Prove that any subset of $\mathbb{R}$ that is not an interval is disconnected.

These results together prove:

Theorem (Connected subsets of $\mathbb{R}$). A subset of $\mathbb{R}$ is a connected if and only if it is an interval.

4. Suppose that $\{A_i\}_{i\in I}$ is a collection of path-connected subsets of a topological space $(X, \mathcal{T})$. Show that, if the intersection $\bigcap_{i\in I} A_i$ is nonempty, then the union $\bigcup_{i\in I} A_i$ is path-connected.

Reflection

(Do this once for each time you present at the board in class, as part of your "participation" grade.)

On a separate piece of paper from your homework solutions, write

- your name;
- the day, worksheet number, and question number(s) you presented to the class;
- a brief reflection (eg, a few sentences) on something that you thought went well in your presentation, and/or something you want to do differently next time. These can be (for example) related to choice of math content, organization of the solution, clarity of the explanation, visual aids, boardwork, presentation delivery like speech and body language, or your responses to audience questions.

Your reflection will be graded just for completion, based on a good-faith effort to think critically about your presentation.