1. **(Quantifier review).** Let P be some mathematical assertion. Explain the difference between the statements

   "For all $n \in \mathbb{N}$ there exists some $N \in \mathbb{N}$ such that P holds."

   "There exists some $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, P holds."

   In particular, interpret the following two statements. Re-phrase both statements in colloquial English, and explain why one statement is true and the other is false.

   "For all $n \in \mathbb{N}$ there exists some $N \in \mathbb{N}$ such that $n > N$."

   "There exists some $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n > N$."

2. **(Functions review).** Let X and Y be sets, and let $f : X \to Y$ be a function.
   (a) Prove or give a counterexample: If $U, V \subseteq Y$ are disjoint, then $f^{-1}(U)$ and $f^{-1}(V)$ in X are disjoint.
   (b) Prove or give a counterexample: If $U, V \subseteq X$ are disjoint, then $f(U)$ and $f(V)$ in Y are disjoint.

3. Let (X, d) be a metric space, and let $(a_n)_{n\in\mathbb{N}}$ be a sequence in X. Show that the following two definitions of convergence are equivalent.

   Definition (Convergence to a_∞). The sequence $(a_n)_{n\in\mathbb{N}}$ converges to a_∞ if for any $\epsilon > 0$, there is some N in $\mathbb{N}$ such that $a_n \in B_\epsilon(x)$ for all $n \geq N$.

   Definition (Convergence to a_∞). The sequence $(a_n)_{n\in\mathbb{N}}$ converges to a_∞ if for any $\epsilon > 0$, there is some $\tilde{N}$ in $\mathbb{N}$ such that $a_n \in B_\epsilon(x)$ for all $n > \tilde{N}$.

4. Let (X, d) be a metric space. Suppose that $S \subseteq T \subseteq X$.
   (a) Suppose that x is an accumulation point of S. Must x be an accumulation point of T? Give a proof or a counterexample.
   (b) Suppose that x is an accumulation point of T. Must x be an accumulation point of S? Give a proof or a counterexample.

5. Let $A \subseteq \mathbb{R}$ be a nonempty bounded subset. Prove that its supremum $\sup(A)$ is either in the set A, or it is an accumulation point of A.

6. Let (X, d) be a metric space, and let $S \subseteq X$ be a sequentially compact subset.
   (a) Show by example that not every subsequence of a sequence $(a_n)_{n\in\mathbb{N}}$ in S need necessarily converge.
   (b) Prove that no subsequence of $(a_n)_{n\in\mathbb{N}}$ can converge to a point in $X \setminus S$.

7. Find an explicit homeomorphism bewteen the intervals $(0, 1)$ and $(1, \infty)$ with the Euclidean metric.

8. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and let $f : X \to Y$ be an open map. Prove or give a counterexample: Then $f(C)$ is closed for every closed subset of X.

9. Let $(X, \mathcal{T})$ be a topological space.
   (a) Suppose that $(X, \mathcal{T})$ is Hausdorff. Let $x \in X$. Show that the intersection of all open sets containing x is equal to $\{x\}$.
   (b) Show that the converse statement does not hold. Specifically, suppose $(X, \mathcal{T})$ is a infinite set with the cofinite topology. Show that $(X, \mathcal{T})$ is not Hausdorff, but for any $x \in X$ the intersection of all open sets containing X is equal to $\{x\}$.

10. Let (X, d) be a metric space, and let $\mathcal{B}$ be a basis for the topology $\mathcal{T}_d$ induced by d.
    (a) Let $S \subseteq X$ be a subset, and $s \in S$. Show that s is an interior point of S if and only if there is some element $B \in \mathcal{B}$ such that $s \in B$ and $B \subseteq S$.
    (b) Deduce that $\mathring{S} = \displaystyle\bigcup_{B \in \mathcal{B}, B \subseteq S} B$.

11. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Let $\mathcal{B}_X$ be a basis for the topology on X, and $\mathcal{B}_Y$ be a basis for the topology on Y. Prove or disprove: the set

$$\{ \, U \times V \mid U \in \mathcal{B}_X, V \in \mathcal{B}_Y \, \}$$

is a basis for the product topology $\mathcal{T}_{X \times Y}$.

12. Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$, and $(Z, \mathcal{T}_Z)$ be topological spaces. Let $f : Z \to X$and $g : Z \to Y$ be continuous functions. Show that the function

$$\begin{aligned} f \times g : Z &\to X \times Y \\ (f \times g)(z) &= (f(z), g(z)) \end{aligned}$$

is continuous with respect to the product topology on $X \times Y$.

13. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Prove or disprove the following statements.
    (a) If X and Y both have the discrete topology, then the product topology on $X \times Y$ is the discrete topology.
    (b) If X and Y both have the indiscrete topology, then the product topology on $X \times Y$ is the indiscrete topology.

14. Let X be a topological space with the indiscrete topology.
    (a) Describe all closed subsets of X.
    (b) Suppose X contains more than one point. Show that X is not metrizable.
    (c) Show that X is compact.
    (d) Show that X is path-connected and connected.

(e) Show that any sequence in X converges to every point of X. Conclude in particular that X is sequentially compact.

(f) Let $A \subsetneq X$ be a proper subset. Show that the interior of A is $\varnothing$.

(g) Let $A \subseteq X$ be a nonempty subset. Show that the closure of A is X.

(h) Let $A \subseteq X$ be subset of X. When is it true that every point of X is an accumulation point of A? When is it true that every point of $X \setminus A$ is an accumulation point of A?

15. Recall that Sierpiński space $\mathbb{S}$ is the set $\mathbb{S} = \{0, 1\}$ with the topology $\Big\{\varnothing, \{1\}, \{0, 1\}\Big\}$.

(a) Show that $\mathbb{S}$ is not Hausdorff and not regular.

(b) Consider $\mathbb{R}$ with the Euclidean metric. Show that every continuous function $\mathbb{S} \to \mathbb{R}$ is constant.

(c) There are 4 possible functions $\mathbb{S} \to \mathbb{S}$. Determine which of these maps are continuous, and which are not continous.

(d) Find all possible homeomorphisms $\mathbb{S} \to \mathbb{S}$.

(e) Show that $\mathbb{S}$ is path-connected and connected.

(f) Show that $\mathbb{S}$ and all of its subsets are compact.

(g) Show that every sequence in $\mathbb{S}$ converges to 0. Under what conditions will a sequence converge to 1?

(h) Find all possible bases for $\mathbb{S}$.

(i) Let $(X, \mathcal{T})$ be a topological space. Show that $U \subseteq X$ is open if and only if the following map is continuous.

$$\chi_U : X \longrightarrow \mathbb{S}$$
$$\chi_U(x) = \begin{cases} 1, & x \in U \\ 0, & x \notin U. \end{cases}$$

16. Let $(X, \mathcal{T})$ be a topological space, and let $A, B \subseteq X$. Suppose $A \cup B$ and $A \cap B$ are connected. Prove that if A and B are both closed or both open, then A and B are connected.

17. Let $(X, \mathcal{T})$ be a topological space, and let $A, B \subseteq X$.

(a) Show that $\mathring{A} \cup \mathring{B}$ is contained in the interior of $A \cup B$.

(b) Show by example that $\mathring{A} \cup \mathring{B}$ need not be equal to the interior of $A \cup B$.

18. Let $(X, \mathcal{T})$ be a compact topological space, and let $f : \mathbb{R} \to X$ be a continuous and closed map. Show that there exists $x \in X$ such that $f^{-1}(x)$ is infinite.

19. Let $X = \{0, 1, 2, 3\}$ be the topological space with the topology $\mathcal{T} = \{\varnothing, \{0, 1\}, \{2, 3\}, X\}$. Show that X is regular, but not Hausdorff.

20. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and let $f : X \to Y$ be a function. Prove that f is open if and only if $f(\mathring{A}) \subseteq \overset{\circ}{f(A)}$ for all sets $A \subseteq X$.

21. Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq B \subseteq X$. Prove that the subspace topology on A (as a subset of X) is the same as the subspace topology on A as a subset of B (with the subspace topology $\mathcal{T}_B$).

22. Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq B \subseteq X$. Prove that A is compact as a subset of B (ie, with respect to the subspace topology $\mathcal{T}_B$ on B) if and only if it is compact as a subset of X.

23. Let $(X, \mathcal{T}_X)$ be a compact topological space, and let $(Y, \mathcal{T}_Y)$ be a Hausdorff topological space. Let $f : X \to Y$ be a continuous map. Show that f is a *closed map*, that is, $f(C) \subseteq Y$ is closed whenever $C \subseteq X$ is closed.

24. **Definition (Adherent sets).** Let $(X, \mathcal{T}_X)$ be a topological space, and let $A, B \subseteq X$. Then A and B are called *adherent* if

$$(A \cap \overline{B}) \cup (\overline{A} \cap B) \neq \varnothing.$$

(a) Give examples of disjoint adherent subsets of $\mathbb{R}$ (with the Euclidean metric).

(b) Let $(X, \mathcal{T}_X)$ be a topological space and $A, B, C \subseteq X$. Prove or give a counterexample: if A and B are adherent, and B and C are adherent, then A and C are adherent.

(c) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and let $f : X \to Y$ be a continuous map. Prove that, if A and B are adherent subsets of X, then $f(A)$ and $f(B)$ are adherent subsets of Y.

(d) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be Hausdorff topological spaces. Suppose that $f : X \to Y$ has the property that, whenever A and B are adherent subsets of X, then $f(A)$ and $f(B)$ are adherent subsets of Y. Prove that f is continuous.

25. **Definition (Dense subsets; nowhere dense subsets).** Let $(X, \mathcal{T})$ be a topological space. A subset $A \subseteq X$ is called *dense* if $\overline{A} = X$. The subset A is called *nowhere dense* if the interior of $\overline{A}$ is empty.

(a) Give an example of a subset of $\mathbb{R}$ that is dense, and a subset of $\mathbb{R}$ that is nowhere dense.

(b) Give an example of a subset of $\mathbb{R}$ that is neither dense nor nowehere dense.

(c) Show that $\mathbb{Q}^n$ is a dense subset of $\mathbb{R}^n$.

(d) Is there a nonempty topological space $(X, \mathcal{T})$ and subset $A \subseteq X$ that is both dense and nowhere dense?

(e) Let A be a dense subset of a topological space $(X, \mathcal{T})$. Show that any open subset $U \subseteq X$ satisfies $\overline{U \cap A} = \overline{U}$.

(f) Let $(X, \mathcal{T})$ be a topological space. Show that $A \subseteq X$ is nowhere dense if and only if $X \setminus \overline{A}$ is a dense open subset of X.

(g) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and let $A \subseteq X$ be a dense subset. Suppose that Y is Hausdorff. Show that a continuous map $f : X \to Y$ is completely determined by its values on A.

26. **Definition (Equivalent Metrics).** Let X be a set. Two metrics d_1 and d_2 on X are called *equivalent* if they both induce the same topology on X.

(a) For a metric d on the set X, use the notation $B_r^d(x)$ to denote the open ball of radius r about a point x. Show that the metrics d_1 and d_2 are equivalent if and only if open balls "nest" in the following sense: for any point $x \in X$ and radius $r > 0$, there exist radii $r_1, r_2 > 0$ such that

$$B_{r_1}^{d_1}(x) \subseteq B_r^{d_2}(x) \qquad \text{and} \qquad B_{r_2}^{d_2}(x) \subseteq B_r^{d_1}(x).$$

(b) Prove that the metrics d_1 and d_2 are equivalent if and only if the identity function $(X, d_1) \to (X, d_2)$ on X and the identity function $(X, d_2) \to (X, d_1)$ are both continuous.

(c) Suppose that, for each $x \in X$, there exist constants $\alpha, \beta > 0$ such that, for every point $y \in X$,

$$\alpha\, d_1(x, y) \leq d_2(x, y) \leq \beta\, d_1(x, y).$$

Show that d_1 and d_2 are equivalent metrics.

(d) Let (X, d) be a metric space. For fixed $\epsilon > 0$, define

$$d_\epsilon(x, y) = \frac{d(x, y)}{\epsilon}.$$

Prove that d_ϵ is a metric on X, and that it is equivalent to d.

(e) Let (X, d) be a metric space. Define

$$d_b(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Prove that d_b is a metric on X, and that it is equivalent to d.

(f) Prove that the following metrics are $\mathbb{R}^2$ are all equivalent.

$$\begin{aligned} d_1((x_1, y_1), (x_2, y_2)) &= |x_1 - x_2| + |y_1 - y_2| \\ d_2((x_1, y_1), (x_2, y_2)) &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ d_\infty((x_1, y_1), (x_2, y_2)) &= \max\Big(|x_1 - x_2|, |y_1 - y_2|\Big) \end{aligned}$$

(In fact, the analogous metrics on $\mathbb{R}^n$ are all equivalent).

27. **Definition (Closure Operator).** Given a set X, let $\mathcal{P}(X)$ denote the collection of all subsets of X. A *closure operator* on X is a function $C : \mathcal{P}(X) \to \mathcal{P}(X)$ obeying the following *Kuratowski* axioms:

   1. $A \subseteq C(A)$ for every subset $A \subseteq X$.
   2. $C(A) = C(C(A))$ for every subset $A \subseteq X$.
   3. $C(\varnothing) = \varnothing$
   4. $C(A \cup B) = C(A) \cup C(B)$ for every $A, B \subseteq X$.

   (a) Let $\mathcal{T}_X$ be any topology on X. Show that the function $A \mapsto \overline{A}$ is a closure operator.

   (b) **(Challenge).** Let X be a set and C a closure operator on X. Prove that there exists a topology on X for which $C(A) = \overline{A}$ for all $A \subseteq X$, and that this topology is unique.

28. **(Challenge.)** Let $(X, \mathcal{T})$ be a Hausdorff topological space with the property that $X \setminus A$ is connected for every finite set A.

   (a) Show that the *configuration space of* X

   $$F_n(X) = \{(x_1, x_2, \dots, x_n) \mid x_i \neq x_j \text{ for every } i \neq j\}$$

   is connected.

   (b) Show that $\mathbb{R}$ does not have this property, but that $\mathbb{R}^n$ does have this property for all $n \geq 2$.