1. Let (X, d) be a metric space. Show that the function $D : X \times X \to \mathbb{R}$

$$D(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

defines a new metric on X.

2. Let (X, d) be a metric space, and let $U \subseteq X$ be an open set. Prove that U is equal to a (possibly infinite) union of open balls of the form $B_r(x) \subseteq X$.

3. Let (X, d) be a metric space. Either prove the following statement, or provide (with proof) a counterexample: For $x_0 \in X$ and $r > 0$ in $\mathbb{R}$, the closure of the open ball $B_r(x_0)$ is the set $\{x \mid d(x_0, x) \leq 1\}$.

4. Let (X, d) be a metric space, and let $S \subseteq X$ be a set with no accumulation points. Prove that S is closed.

5. Consider the real numbers $\mathbb{R}$ with the Euclidean metric. Let $S = (0, 1) \subseteq \mathbb{R}$. Give a complete and rigorous proof of what the sets $\mathring{S}$, $\overline{S}$, and ∂S are.

6. Recall our definition of a bounded set:

   Definition (Bounded subset.) Let (X, d) be a metric space. A subset $S \subseteq X$ is called *bounded* if there is some $x_0 \in X$ and some $R \in \mathbb{R}$ with $R > 0$ such that $S \subseteq B_R(x_0)$.

   Let (X, d) be a metric space and let $S \subseteq X$ be a bounded set in X. Show that if x is *any* point of X, then there is some $\widetilde{R} > 0$ such that $B_{\widetilde{R}}(x)$ contains S.

7. Let (X, d) be a metric space, and let $S \subseteq X$ be a **finite** subset of X. Give a rigorous proof that S . . .

   (a) is closed.

   (b) is bounded.

   (c) has no accumulation points.

8. Let (X, d) be a metric space, and let $S \subseteq X$ be a **finite** subset of X. Is it necessarily true that $\mathring{S} = \varnothing$?

9. Let A be a subset of a metric space. Starting from the definition of *open* and *boundary* (instead of quoting results from the homework), prove the following: If A is an open set, then

$$A \cap \partial A = \varnothing.$$

10. Suppose that (X, d) is a metric space, and that $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are Cauchy sequences in X. Show that the sequence of real numbers $d(x_n, y_n)$ is a Cauchy sequence in $\mathbb{R}$ (with the Euclidean metric).

11. Suppose that (X, d) is a metric space with the property that every bounded sequence converges. Prove that X is a single point.

12. Let X be a nonempty set, endowed with the discrete metric. Rigorously prove or disprove the following. Is X necessarily a complete metric space?

13. (a) Prove that $\mathbb{R}$ with the Euclidean metric is a complete metric space.

    (b) Prove that the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ with the Euclidean metric is **not** a complete metric space.

    (c) The function

$$f : \mathbb{R} \longrightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$
$$f(x) = \arctan(x)$$

    is continuous (which you do not need to prove). Use this function to show that the continuous image of a complete metric space need not be complete.

    (d) Suppose that (X, d_X) and (Y, d_Y) are metric spaces, and that X a complete metric space. Suppose that $f : X \to Y$ is a continuous map satisfying

$$d_Y\Big(f(x_1), f(x_2)\Big) = d_X(x_1, x_2) \qquad \text{for all } x_1, x_2 \in X.$$

    Show that $f(X)$ is a complete metric space.

14. Let (X, d_X) and (Y, d_Y) be metric spaces. Recall from Homework 5 that $X \times Y$ is then a metric space with metric

$$d_{X\times Y} : \ (X \times Y) \times (X \times Y) \longrightarrow \mathbb{R}$$
$$d_{X\times Y}((x_1, y_1), (x_2, y_2)) = \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2}.$$

    (a) Suppose that $(x_n)_{n\in\mathbb{N}}$ is a sequence in X converging to a point $x \in X$, and that $(y_n)_{n\in\mathbb{N}}$ is a sequence in Y converging to a point $y \in Y$. Prove that the sequence $\Big((x_n, y_n)\Big)_{n\in\mathbb{N}}$ in $X \times Y$ is convergent, and its limit is

$$\lim_{n\in\mathbb{N}} (x_n, y_n) = (x, y).$$

    (b) Suppose that $\Big((x_n, y_n)\Big)_{n\in\mathbb{N}}$ in $X \times Y$ is a convergent sequence with

$$\lim_{n\in\mathbb{N}} (x_n, y_n) = (x, y).$$

    Does it follow that the sequence $(x_n)_{n\in N}$ in X converges to x?

    (c) Prove that X and Y are sequentially compact if and only if $X \times Y$ is sequentially compact.

15. Let (X, d_X) and (Y, d_Y) be metric spaces. Define

$$d_1 : \ (X \times Y) \times (X \times Y) \longrightarrow \mathbb{R}$$
$$d_1((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2).$$

(a) Show that d_1 is a metric on $X \times Y$.

(b) Show that if $U \subseteq X$ and $V \subseteq Y$ are open sets, then $U \times V$ is an open subset of $X \times Y$.

16. Let X be a finite set (of, say, n elements), and let d be a metric on X. What is the topology $\mathcal{T}_d$ on X induced by d? Show in particular that this topology will be the same for every possible metric d.

17. Let X be a set, and suppose that $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on X.

    (a) Show that the intersection $\mathcal{T}_1 \cap \mathcal{T}_2$ is a topology on X.

    (b) Show by example that the union $\mathcal{T}_1 \cup \mathcal{T}_2$ need not be a topology on X.

18. (a) Show that the following collection of subsets of $\mathbb{R}$ forms a topology on $\mathbb{R}$.

$$\mathcal{T} = \{\varnothing, \mathbb{R}\} \cup \left\{ (b, \infty) \;\middle|\; b \in \mathbb{R} \right\}.$$

    (b) Determine whether the topology $\mathcal{T}$ is metrizable.