Final Exam

Math 490
19 December 2018
Jenny Wilson

Name: ____________________

Instructions: This exam has 4 questions for a total of 40 points.

Each student may bring in one double-sided ($8\frac{1}{2}$" $\times$ 11") sheet of notes, which they must have either hand-written or typed (in font size at least 12) themselves.

The exam is closed-book. No books, additional notes, cell phones, calculators, or other devices are permitted. Scratch paper is available.

Fully justify your answers unless otherwise instructed. You may cite any results proved in class, on a quiz, or on the homeworks without proof.

You have 120 minutes to complete the exam. If you finish early, consider checking your work for accuracy.

Jenny is available to answer questions.

Question	Points	Score
1	28	
2	4	
3	4	
4	4	
Total:	40	

1. (28 points) For each of the following statements: if the statement is true, write "`True`". If the statement is not true, state a counterexample. No further justification needed.

   (i) Let (X, d) be a metric space. Then the union of an arbitrary collection of closed sets in X is closed.

   (ii) Let (X, d) be a metric space, and $S \subseteq X$ a **finite** set. Then $\mathring{S} = \varnothing$.

   (iii) Consider $\mathbb{Z}$ as a metric space with the Euclidean metric. Then every subset $S \subseteq \mathbb{Z}$ is both open and closed.

   (iv) Let A be a subset of a metric space (X, d), and let $(a_n)_{n\in\mathbb{N}}$ be a sequence of points in A that converges to an element $a_\infty \in X$. Then $a_\infty \in \overline{A}$.

   (v) Let A be a subset of a metric space (X, d). Then any element of ∂A must be both an accumulation point of A, and an accumulation point of $X \setminus A$.

   (vi) Let $(X, \mathcal{T})$ be a topological space, and let $x \in X$. Let $(a_n)_{n\in\mathbb{N}}$ be the constant sequence $x\ x\ x\ x\ x\ \cdots$. Then $(a_n)_{n\in\mathbb{N}}$ converges to x.

   (vii) Let $(X, \mathcal{T})$ be a topological space, and let x, y be **distinct** points in X. Let $(a_n)_{n\in\mathbb{N}}$ be the sequence $x\ y\ x\ y\ x\ y\ x\ y \cdots$. Then $(a_n)_{n\in\mathbb{N}}$ does not converge.

(viii) Let $(X, \mathcal{T})$ be a topological space, and let $x \in X$. Then the set $\{x\} \subseteq X$ is closed.

(ix) Let $(X, \mathcal{T})$ be a topological space, and $A \subseteq X$ a subset. Then $\overline{X \setminus A} = X \setminus \overline{A}$.

(x) Let $(X, \mathcal{T})$ be a topological space, and let $S \subseteq X$ be a subset. If X is Hausdorff, then the subspace topology on S is Hausdorff.

(xi) Let $(X, \mathcal{T})$ be a disconnected topological space. Then there is some proper nonempty subset $A \subseteq X$ that is both open and closed.

(xii) Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$ be a subset. If A is connected, then $\overline{A}$ is connected.

(xiii) Let A and B be nonempty subsets of $\mathbb{R}$ (with the Euclidean metric). If $A \cap B = \varnothing$, then $A \cup B$ is disconnected.

(xiv) Let A and B be nonempty subsets of a topological space $(X, \mathcal{T})$. If A and B are connected and $A \cap B$ is nonempty, then $A \cap B$ is connected.

(xv) Suppose that (X, d) is a compact metric space. Then X is bounded.

(xvi) Let $(X, \mathcal{T})$ be a compact topological space. Then every closed subset of X is compact.

(xvii) Let $(X, \mathcal{T})$ be a compact topological space. Then every compact subset of X is closed in X.

(xviii) Consider $[0, 1]$ with the Euclidean metric. Then any countably infinite subset $\{ a_n \mid n \in \mathbb{N} \} \subseteq [0, 1]$ is compact.

(xix) Consider $[0, 1]$ with the Euclidean metric. Then any countably infinite subset $\{ a_n \mid n \in \mathbb{N} \} \subseteq [0, 1]$ is non-compact.

(xx) Let (X, d_X) and (Y, d_Y) be metric spaces, and let $A \subseteq X$ and $B \subseteq Y$ be compact subsets. Then $A \times B$ is a compact subset of $X \times Y$ (with the product topology).

(xxi) Consider a Hausdorff topological space $(Y, \mathcal{T})$ and $\mathbb{R}$ with the Euclidean metric. Let $f : \mathbb{R} \to Y$ be a continuous function. Then f is completely determined by its values on $\mathbb{Q} \subseteq \mathbb{R}$.

(xxii) Let (X, d_X) and (Y, d_Y) be metric spaces, and $f : X \to Y$ a continuous function. If $B \subseteq X$ is bounded, then $f(B)$ is bounded.

(xxiii) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and $f : X \to Y$ a continuous function. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in X. If $(a_n)_{n\in\mathbb{N}}$ converges, then the sequence $(f(a_n))_{n\in\mathbb{N}}$ in Y converges.

(xxiv) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and $f : X \to Y$ a continuous function. If X has the discrete topology, then so does the subspace $f(X) \subseteq Y$.

(xxv) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and $f : X \to Y$ a continuous function. If the subspace $f(X) \subseteq Y$ has the discrete topology, then so does X.

(xxvi) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and $f : X \to Y$ a continuous function. If X is path-connected, then $f(X)$ is path-connected.

(xxvii) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and $f : X \to Y$ a continuous function. If $C \subseteq Y$ is compact, then $f^{-1}(C)$ is compact.

(xxviii) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and $f : X \to Y$ a continuous function. If X is Hausdorff, then $f(X)$ is Hausdorff.

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2. (4 points) Let $(X, \mathcal{T}_X)$ be a topological space with basis $\mathcal{B}$. Show that X is compact if and only if every cover of X consisting of elements of $\mathcal{B}$ has a finite subcover.

3. (4 points) Let (X, d) be a metric space with at least two elements. Show that there exist nonempty open sets in X whose closures are disjoint.

4. (4 points) Let $(X, \mathcal{T}_X)$ be a topological space, and let $X \times X$ be a topological space with the product topology $\mathcal{T}_{X \times X}$. The set

$$\Delta = \{ \ (x, x) \mid x \in X \ \} \ \subseteq \ X \times X$$

is called the *diagonal* of $X \times X$. Prove that X is Hausdorff if and only if the diagonal Δ is a closed subset of $X \times X$.