Final Exam

Math 490
19 December 2018
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Name: ____

Instructions: This exam has 4 questions for a total of 40 points.

Each student may bring in one double-sided ($8\frac{1}{2}$" $\times$ 11") sheet of notes, which they must have either hand-written or typed (in font size at least 12) themselves.

The exam is closed-book. No books, additional notes, cell phones, calculators, or other devices are permitted. Scratch paper is available.

Fully justify your answers unless otherwise instructed. You may cite any results proved in class, on a quiz, or on the homeworks without proof.

You have 120 minutes to complete the exam. If you finish early, consider checking your work for accuracy.

Jenny is available to answer questions.

Question	Points	Score
1	28	
2	4	
3	4	
4	4	
Total:	40	

1. (28 points) For each of the following statements: if the statement is true, write "`True`". If the statement is not true, state a counterexample. No further justification needed.

   (i) Let (X, d) be a metric space. Then the union of an arbitrary collection of closed sets in X is closed.

   False. For example, let $X = \mathbb{R}$ with the Euclidean metric. A single point $\{x\} \subseteq \mathbb{R}$ is closed, but the union $(0,1) = \bigcup_{x \in (0,1)} \{x\}$ in $\mathbb{R}$ is not closed.

   (ii) Let (X, d) be a metric space, and $S \subseteq X$ a **finite** set. Then $\mathring{S} = \varnothing$.

   False. For example, consider $X = \mathbb{N}$ with the discrete metric, and $S \subseteq \mathbb{N}$ any finite nonempty set (say, $S = \{1, 2, 3\}$). Then $\mathring{S} = S$ is nonempty.

   (iii) Consider $\mathbb{Z}$ as a metric space with the Euclidean metric. Then every subset $S \subseteq \mathbb{Z}$ is both open and closed.

   True.

   (iv) Let A be a subset of a metric space (X, d), and let $(a_n)_{n \in \mathbb{N}}$ be a sequence of points in A that converges to an element $a_\infty \in X$. Then $a_\infty \in \overline{A}$.

   True.

   (v) Let A be a subset of a metric space (X, d). Then any element of ∂A must be both an accumulation point of A, and an accumulation point of $X \setminus A$.

   False. For example, consider $X = \mathbb{R}$ with the Euclidean metric, and $A = \{1\}$. Then $\partial A = \{1\}$ but 1 is not an accumulation point of A.

   (vi) Let $(X, \mathcal{T})$ be a topological space, and let $x \in X$. Let $(a_n)_{n \in \mathbb{N}}$ be the constant sequence $x\ x\ x\ x\ x\ \cdots$. Then $(a_n)_{n \in \mathbb{N}}$ converges to x.

   True.

   (vii) Let $(X, \mathcal{T})$ be a topological space, and let x, y be **distinct** points in X. Let $(a_n)_{n \in \mathbb{N}}$ be the sequence $x\ y\ x\ y\ x\ y\ x\ y \cdots$. Then $(a_n)_{n \in \mathbb{N}}$ does not converge.

   False. For example, consider $X = \{x, y\}$ with the indiscrete topology. Then the sequence converges to both x and to y.

(viii) Let $(X, \mathcal{T})$ be a topological space, and let $x \in X$. Then the set $\{x\} \subseteq X$ is closed.

False. For example, consider $X = \{x, y\}$ with the indiscrete topology. Then $\{x\}$ is not closed.

(ix) Let $(X, \mathcal{T})$ be a topological space, and $A \subseteq X$ a subset. Then $\overline{X \setminus A} = X \setminus \overline{A}$.

False. For example, $A = [0, \infty)$ as a subset of $X = \mathbb{R}$ with the Euclidean metric. Then $0 \in \overline{X \setminus A} = (-\infty, 0]$ but $0 \notin X \setminus \overline{A} = (-\infty, 0)$.

(x) Let $(X, \mathcal{T})$ be a topological space, and let $S \subseteq X$ be a subset. If X is Hausdorff, then the subspace topology on S is Hausdorff.

True.

(xi) Let $(X, \mathcal{T})$ be a disconnected topological space. Then there is some proper nonempty subset $A \subseteq X$ that is both open and closed.

True.

(xii) Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$ be a subset. If A is connected, then $\overline{A}$ is connected.

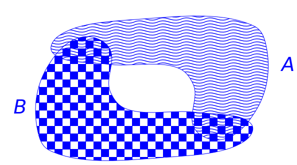
True.

(xiii) Let A and B be nonempty subsets of $\mathbb{R}$ (with the Euclidean metric). If $A \cap B = \varnothing$, then $A \cup B$ is disconnected.

False. For example, consider $A = (-\infty, 0)$ and $B = [0, \infty)$ as subsets of $\mathbb{R}$ with the Euclidean metric. Then A and B are disjoint but $A \cup B = \mathbb{R}$ is connected.

(xiv) Let A and B be nonempty subsets of a topological space $(X, \mathcal{T})$. If A and B are connected and $A \cap B$ is nonempty, then $A \cap B$ is connected.

False. For example, consider the following two subsets of $\mathbb{R}^2$ (with the Euclidean metric).



(xv) Suppose that (X, d) is a compact metric space. Then X is bounded.

True.

(xvi) Let $(X, \mathcal{T})$ be a compact topological space. Then every closed subset of X is compact.

True.

(xvii) Let $(X, \mathcal{T})$ be a compact topological space. Then every compact subset of X is closed in X.

False. For example, consider the set $X = \{1, 2\}$ with the indiscrete metric. Then the set $\{1\}$ is compact but not closed.

(xviii) Consider $[0, 1]$ with the Euclidean metric. Then any countably infinite subset $\{ a_n \mid n \in \mathbb{N} \} \subseteq [0, 1]$ is compact.

False. For example, the set $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ is not compact. (Since it does not contain its accumulation point 0, it is not closed; compact subsets of $\mathbb{R}$ are closed and bounded).

(xix) Consider $[0, 1]$ with the Euclidean metric. Then any countably infinite subset $\{ a_n \mid n \in \mathbb{N} \} \subseteq [0, 1]$ is non-compact.

False. For example, the set $\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ is compact (since it is a closed and bounded subset of $\mathbb{R}$).

(xx) Let (X, d_X) and (Y, d_Y) be metric spaces, and let $A \subseteq X$ and $B \subseteq Y$ be compact subsets. Then $A \times B$ is a compact subset of $X \times Y$ (with the product topology).

True.

(xxi) Consider a Hausdorff topological space $(Y, \mathcal{T})$ and $\mathbb{R}$ with the Euclidean metric. Let $f : \mathbb{R} \to Y$ be a continuous function. Then f is completely determined by its values on $\mathbb{Q} \subseteq \mathbb{R}$.

True.

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(xxii) Let (X, d_X) and (Y, d_Y) be metric spaces, and $f : X \to Y$ a continuous function. If $B \subseteq X$ is bounded, then $f(B)$ is bounded.

False. For example, consider $X = Y = (0, \infty)$ in the Euclidean metric, and the continuous map $f : (0, \infty) \to (0, \infty)$ given by $f(x) = \frac{1}{x}$. Then $(0, 1) \subseteq (0, \infty)$ is bounded, but $f\Big((0, 1)\Big) = (1, \infty)$ is not bounded.

(xxiii) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and $f : X \to Y$ a continuous function. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in X. If $(a_n)_{n\in\mathbb{N}}$ converges, then the sequence $(f(a_n))_{n\in\mathbb{N}}$ in Y converges.

True.

(xxiv) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and $f : X \to Y$ a continuous function. If X has the discrete topology, then so does the subspace $f(X) \subseteq Y$.

False. Let $X = \mathbb{R}$ with the discrete topology and $Y = \mathbb{R}$ with the indiscrete topology. Let $f : X \to Y$ be the identity map on $\mathbb{R}$. Then f is continuous and X has the discrete topology, but $f(X) = \mathbb{R}$ does not.

(xxv) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and $f : X \to Y$ a continuous function. If the subspace $f(X) \subseteq Y$ has the discrete topology, then so does X.

False. Let $X = \mathbb{R}$ with the Euclidean metric, and let $Y = \{0\}$ be a single point. Let $f : X \to Y$ be the constant map $f(x) = 0$. Then $f(X) = Y = \{0\}$ has the discrete topology, but $X = \mathbb{R}$ does not.

(xxvi) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and $f : X \to Y$ a continuous function. If X is path-connected, then $f(X)$ is path-connected.

True.

(xxvii) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and $f : X \to Y$ a continuous function. If $C \subseteq Y$ is compact, then $f^{-1}(C)$ is compact.

False. Let $X = \mathbb{R}$ with the Euclidean metric, and let $Y = \{0\}$ be a single point. Let $f : X \to Y$ be the constant map $f(x) = 0$. Then f is continuous, and $C = \{0\}$ is compact, but $f^{-1}(C) = \mathbb{R}$ is not compact.

(xxviii) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and $f : X \to Y$ a continuous function. If X is Hausdorff, then $f(X)$ is Hausdorff.

False. Let $X = \mathbb{N}$ with the discrete topology and $Y = \mathbb{N}$ with the indiscrete topology. Let $f : X \to Y$ be the identity map on $\mathbb{N}$. Then f is continuous and X is Hausdorff, but $f(X) = Y$ is not Hausdorff.

2. (4 points) Let $(X, \mathcal{T}_X)$ be a topological space with basis $\mathcal{B}$. Show that X is compact if and only if every cover of X consisting of elements of $\mathcal{B}$ has a finite subcover.

Solution. First suppose that X is compact. Then any cover of X consisting of elements of $\mathcal{B}$ is, in particular, an open cover. By definition of compactness, it must have a finite subcover.

Now suppose conversely that X is a space with the property that every open cover of X consisting of elements of $\mathcal{B}$ has a finite subcover. We wish to show that X is compact, that is, **any** open cover of X has a finite subcover.

So let $\mathcal{U}$ be an open cover of X. By definition of a basis, each open set $U \in \mathcal{U}$ can be written as a union of basis elements $U = \cup_{i \in I_U} B_{i,U}$ with $B_{i,U} \in \mathcal{B}$. So consider the new open cover

$$\mathcal{C} = \{B_{i,U} \mid U \in \mathcal{U}, i \in I_U\}.$$

To see that $\mathcal{C}$ covers X, consider a point $x \in X$. Since $\mathcal{U}$ covers, $x \in U$ for some $U \in \mathcal{U}$. But since $U = \cup_{i \in I_U} B_{i,U}$, $x \in B_{i,U}$ for some $B_{i,U}$. Hence $\mathcal{C}$ is an open cover of X.

Then, since $\mathcal{C}$ consists of basis elements, we know that it has some finite subcover,

$$B_{i_1,U_1}, B_{i_2,U_2}, \ldots, B_{i_n,U_n}.$$

This means that, for any $x \in X$, $x \in B_{i_j,U_j}$ for some j. But $B_{i_j,U_j} \subseteq U_j$, so every $x \in X$ is contained in U_j for some j. It follows that

$$U_1, U_2, \ldots, U_n$$

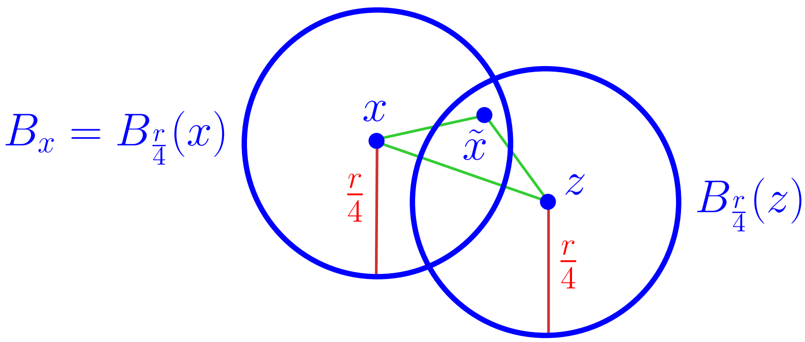
is a finite subcover of $\mathcal{U}$. We conclude that X is compact, as claimed.

3. (4 points) Let (X, d) be a metric space with at least two elements. Show that there exist nonempty open sets in X whose closures are disjoint.

Solution. Suppose that X contains the two distinct elements x and y, and suppose that $d(x, y) = r$. Then $r > 0$ by definition of a metric, and so the sets $B_x = B_{\frac{r}{4}}(x)$ and $B_y = B_{\frac{r}{4}}(y)$ are open balls around x and y, respectively. We will show that these two nonempty open sets have disjoint closure.

Suppose (for the sake of contradiction) that z were an element in $\overline{B_x}$ and $\overline{B_y}$. This means that every open neighbourhood U_z of z contains a point in B_x and contains a point in B_y. So consider the open neighbourhood $U_z = B_{\frac{r}{4}}(z)$.

By assumption this neighbourhood contains a point $\tilde{x} \in B_x$.



But then observe that

$$\begin{aligned} d(x, z) &\leq d(x, \tilde{x}) + d(\tilde{x}, z) \\ &< \frac{r}{4} + \frac{r}{4} \qquad \text{(since } \tilde{x} \in B_{\frac{r}{4}}(x) \text{ and } \tilde{x} \in B_{\frac{r}{4}}(z)\text{)} \\ &= \frac{r}{2} \end{aligned}$$

Since $B_{\frac{r}{4}}(z)$ must also contain a point of B_y, the same argument shows that $d(y, z) < \frac{r}{2}$. But then

$$\begin{aligned} d(x, y) &\leq d(x, z) + d(z, y) \\ &< \frac{r}{2} + \frac{r}{2} \\ &= r \end{aligned}$$

which contradicts our premise that $d(x, y) = r$. Thus no such element z can exist, and we conclude that $\overline{B_x} \cap \overline{B_y} = \varnothing$ as claimed.

4. (4 points) Let $(X, \mathcal{T}_X)$ be a topological space, and let $X \times X$ be a topological space with the product topology $\mathcal{T}_{X\times X}$. The set

$$\Delta = \{ \ (x,x) \mid x \in X \ \} \ \subseteq \ X \times X$$

is called the *diagonal* of $X \times X$. Prove that X is Hausdorff if and only if the diagonal Δ is a closed subset of $X \times X$.

Solution. First, suppose that X is a Hausdorff space. To prove that Δ is closed, we must show that its complement is open. To prove that the complement is open, it suffices to show that every point (x,y) in the complement of Δ has some open neighbourhood contained in the complement of Δ.

So let (x,y) be a point with $(x,y) \notin \Delta$. This means that $x \neq y$. But since X is Hausdorff, by definition, it follows that there are disjoint open sets U and V with $x \in U$ and $y \in V$. We claim that $U \times V$ is the desired open set containing (x,y) and contained in the complement of Δ.

The set $U \times V$ is open in $X \times Y$ by definition of the product topology, and it contains (x,y) by construction. Now, let (u,v) be any point in $U \times V$. Then $u \in U$ and $v \in V$. Since U and V are disjoint, it follows that $u \neq v$, and so $(u,v) \notin \Delta$. This shows that $(U \times V) \cap \Delta = \varnothing$, and so $U \times V$ is contained in the complement of Δ as claimed. Hence Δ is closed.

Conversely, suppose that Δ is closed. Let x and y be distinct elements of X. To show that X is Hausdorff, we wish to find disjoint neighbourhoods about x and y. Since Δ is closed, by definition, the complement $(X \times X) \setminus \Delta$ is open in the product topology. Since the product topology is generated by sets of the form $U \times V$ (with $U, V \subseteq X$ both open), we know that the element $(x,y) \in (X \times X) \setminus \Delta$ must be contained in some open set of the form $U \times V \subseteq (X \times X) \setminus \Delta$.

This means that U is an open neighbourhood of x, and V is an open neighbourhood of y. Moreover, U and V must be disjoint, If not, if $z \in U \cap V$, then $(z,z) \in (U \times V) \cap \Delta$ would contradict our assumption that $U \times V$ is contained in the complement of Δ. Thus, X is Hausdorff as claimed.