Midterm Exam Solutions

Math 490
23 October 2018
Jenny Wilson

Name: ______________________________

Instructions: This exam has 4 questions for a total of 20 points.

The exam is closed-book. No books, notes, cell phones, calculators, or other devices are permitted. Scratch paper is available.

Fully justify your answers unless otherwise instructed. You may quote any results proved in class or on the homeworks without proof.

You have 80 minutes to complete the exam. If you finish early, consider checking your work for accuracy.

Jenny is available to answer questions.

Question	Points	Score
1	10	
2	3	
3	3	
4	4	
Total:	20	

1. (10 points) For each of the following, write down an example (if an example exists), or otherwise write "`Does not exist`". No further justification needed.

   (a) Metric spaces (X, d_X) and (Y, d_Y), a continuous function $f : X \to Y$, and a closed set $C \subseteq Y$ such that the **preimage** $f^{-1}(C)$ is not closed.

   Does not exist.

   (b) Metric spaces (X, d_X) and (Y, d_Y), a continuous function $f : X \to Y$, and an open set $U \subseteq X$ such that the **image** $f(U)$ is not open.

   Example: Let $X = Y = \mathbb{R}$ with the Euclidean metric. Let $f(x) = |x|$, and let $U = (-1, 1)$. Then f is continuous and U is open, but $f(U) = [0, 1)$ is not open.

   (c) A metric space (X, d) and a nonempty set $S \subseteq X$ such that every point of S is an accumulation point of S.

   Example: Let $X = \mathbb{R}$ with the Euclidean metric, and $S = (0, 1)$.

   (d) A metric space (X, d), a subset $S \subseteq X$, and a point $x \in \partial S$ that is not an accumulation point of S.

   Example: Let $X = \mathbb{R}$ with the Euclidean metric, and let $S = \{1\}$. Then $1 \in \partial S$, but 1 is not an accumulation point of ∂S.

   (e) A metric space (X, d) and a sequence $(a_n)_{n \in \mathbb{N}}$ in X that is convergent but not Cauchy.

   Does not exist.

   (f) A metric space (X, d) and a subset $S \subseteq X$ such that ∂S is not closed.

   Does not exist.

   (g) A metric space (X, d) that is closed and bounded, but not sequentially compact.

   Example: Let $X = (0, 1)$ with the Euclidean metric.

   (h) A metric space (X, d) that is sequentially compact, but not bounded.

   Does not exist.

   (i) A topology $(X, \mathcal{T})$, and closed sets C and D in X such that $C \cup D$ is not closed.

   Does not exist.

   (j) A topology $(X, \mathcal{T})$ that is not metrizable.

   Example: Let $X = \{0, 1, 2\}$, and $\mathcal{T} = \{\varnothing, X\}$.

2. (3 points) Let (X, d) be a metric space, and let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in X. Prove that the set $\{a_n \mid n \in \mathbb{N}\}$ is bounded.

Suppose that $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. This means that, for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ so that $d(a_n, a_m) < \epsilon$ for all $n, m \geq N$.

So let $\epsilon = 1$, and fix N such that $d(a_n, a_m) < 1$ for all $n, m \geq N$. Define

$$M = \max_{k=1,\dots,N-1} d(a_k, a_N).$$

This maximum exists since there are only finitely many terms $a_1, \dots, a_{N-1}$. Finally, choose R so that $R > \max(1, M)$. We claim that $\{a_n \mid n \in \mathbb{N}\} \subseteq B_R(a_N)$.

To verify this claim, consider a point a_n. If $n < N$, then

$$d(a_n, a_N) \leq M < R \qquad \text{by the definition of } M,$$

so $a_n \in B_R(a_N)$. If $n \geq N$, then

$$d(a_n, a_N) < 1 < R \qquad \text{by the Cauchy condition,}$$

so again $a_n \in B_R(a_N)$. We conclude that the set $\{a_n \mid n \in \mathbb{N}\}$ is bounded, as desired.

3. (3 points) Let (X, d) be a metric space, and let $A, B \subseteq X$. Show that, if A and B are sequentially compact, then so is $A \cap B$.

Suppose that A is sequentially compact. This means that every sequence in A has a subsequence that converges to a point of A. Assume similarly that B is sequentially compact.

Our goal is to show that $A \cap B$ is sequentially compact. So let $(a_n)_{n\in\mathbb{N}}$ be a sequence of points in $A \cap B$. Since A is sequentially compact, there is some subsequence $(a_{n_i})_{i\in\mathbb{N}}$ converging to a point $a \in A$. But, since B is sequentially compact, we have proved that B is closed. Since $(a_{n_i})_{i\in\mathbb{N}}$ is a convergent sequence in the closed set B, we proved moreover that its limit $a = \lim_{i\in\mathbb{N}} a_{n_i}$ must be in B.

So $a \in A \cap B$. Thus we have found a subsequence $(a_{n_i})_{i\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$ converging to a point of $A \cap B$. We conclude that $A \cap B$ is sequentially compact.

4. (4 points) Let (X,d) be a metric space, and let S be a nonempty subset of X. Prove that a point $x \in X$ is in $\overline{S}$ if and only if there is a sequence of points $(s_n)_{n\in\mathbb{N}}$ in S converging to x.

First suppose that there is a sequence of points $(s_n)_{n\in\mathbb{N}}$ in S converging to x. We proved that this means that, given any neighbourhood U of x, there is some N such that $s_n \in U$ for all $n \geq N$. In particular, given any neighbourhood U of x, there is a point $s_N \in S$ with $s_N \in U$. Thus, by definition, we see that $x \in \overline{S}$.

Suppose conversely that $x \in \overline{S}$. This means that, for any neighbourhood U of x, there is some element $s \in S$ in U. So first consider the neighbourhood $B_1(x)$ of x. It contains some point of S; call this point s_1. Next consider the neighbourhood $B_{\frac{1}{2}}(x)$ of x. It contains some point of S; call this point s_2. (We allow the possibility that $s_1 = s_2$.) In general, for $n \in \mathbb{N}$, let s_n be any element of S in $B_{\frac{1}{n}}(x)$.

We claim that the resulting sequence converges to x. To check this, let $\epsilon > 0$ be given. Then choose $N \in \mathbb{N}$ large enough so that $\frac{1}{N} < \epsilon$. Then, for $n \geq N$ we have $\frac{1}{n} \leq \frac{1}{N} < \epsilon$, so $B_{\frac{1}{n}}(x) \subseteq B_\epsilon(x)$, and we see that $s_n \in B_\epsilon(x)$ for all $n \geq N$. Thus the sequence $(s_n)_{n\in\mathbb{N}}$ converges to x. We conclude that there is a sequence in S converging to x, as desired.