1. (10 points) For each of the following, write down an example (if an example exists), or otherwise write "`Does not exist`". No further justification needed.

   (a) A metric space (X, d), a subset $Y \subseteq X$ viewed as a metric space under the restriction of the metric d, and a subset $U \subseteq Y$ that is open as a subset of Y but not open as a subset of X.

   Example: Let $X = \mathbb{R}$ with the Euclidean metric. Let $Y = \{1\}$. Let $U = Y = \{1\}$. Then U is open in Y, but not in $\mathbb{R}$.

   (b) Metric spaces (X, d_X) and (Y, d_Y), a continuous function $f : X \to Y$, and an open set $U \subseteq Y$ such that $f^{-1}(U)$ is closed.

   Example: Let X and Y both be $\mathbb{R}$ (with the Euclidean metric). Let $f : \mathbb{R} \to \mathbb{R}$ be the identity map. Take $U = \mathbb{R}$. Then U is open, and $f^{-1}(U) = \mathbb{R}$ is closed (as well as open).

   (c) Metric spaces (X, d_X) and (Y, d_Y), a continuous function $f : X \to Y$, and a set $A \subseteq X$ such that $f(\overline{A}) \not\subseteq \overline{f(A)}$.

   Does not exist.

   (d) A metric space (X, d) and a nonempty set $S \subseteq X$ such that S is open, and S has no accumulation points.

   Example: Let $X = \{1\}$ be a single point. Then the set $S = X = \{1\}$ is open in X, but has no accumulation points.

   (e) A nonempty set $S \subseteq \mathbb{R}$ (with the Euclidean metric) such that no point contained in S is an accumulation point of S, but S has an accumulation point $x \notin S$.

   Example: Let $S = \{ \frac{1}{n} \mid n \in \mathbb{N} \}$. Then every point of S is an isolated point, but $0 \in \mathbb{R}$ is an accumulation point of S.

   (f) A sequence of real numbers $(a_n)_{n\in\mathbb{N}}$ that is bounded but does not converge.

   Example: The sequence 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 . . . is bounded but not convergent.

   (g) A sequence of real numbers $(a_n)_{n\in\mathbb{N}}$ that converges but is not bounded.

   Does not exist.

   (h) A metric space (X, d), and a subset $S \subseteq X$ that is sequentially compact but not closed.

   Does not exist.

   (i) A metric space (X, d) that is not complete.

   Example: The metric space $(0, 1)$ with the Euclidean metric is not complete.

   (j) A metric space (X, d) and a sequence $(a_n)_{n\in\mathbb{N}}$ in X such that $\{a_n \mid n \in \mathbb{N}\}$ is closed and bounded, but has no convergent subsequence.

   Example: Let X be an infinite set and let d be the discrete metric on X. Then let $(a_n)_{n\in\mathbb{N}}$ be any sequence such that each term a_n is a distinct element of X. Then $\{a_n \mid n \in \mathbb{N}\}$ is closed and bounded, but has no convergent subsequence.

2. (2 points) Either prove the following statement, or provide (with proof) a counterexample: Let (X, d) be a metric space, $A \subseteq X$ a subset, and $(a_n)_{n \in \mathbb{N}}$ a sequence of points in A that converge to a point a_∞. Then a_∞ is an accumulation point of A.

Counterexample: Let $\mathbb{R}$ be the real numbers. Let $A = \{5\}$. Then consider the constant sequence with $a_n = 5$ for all n. The sequence $(a_n)_{n \in \mathbb{N}}$ is contained in A and converges to the point 5.

The point 5 is not, however, an accumulation point of A. Recall that, for 5 to be an accumulation point of A, every open ball of the form $B_r(5)$ must contain a point in A other than 5. But consider the ball of radius (say) $r = 1$ around 5. There is no point of A in this ball except for 5. We conclude that 5 is not an accumulation point of A.

3. (a) (2 points) Let (X, d) be a metric space, and let $x, y \in X$. Prove that there is an open set $U \subseteq X$ such that $x \in U$ but $y \notin U$.

Let x and y be distinct points of X. Then, by definition of a metric, the distance

$$d(x, y) > 0.$$

Let $r = \frac{1}{2} d(x, y)$. By construction $r > 0$. Let

$$U = B_r(x) = \{ \tilde{x} \in X \mid d(x, \tilde{x}) < r \}.$$

We proved in class that U is open, and moreover that $x \in U$. However, because

$$d(x, y) = 2r > r,$$

the point y is not in U. This proves the claim.

(b) (1 point) Sierpinski space $\mathbb{S}$ is the set $\{0, 1\}$ with the topology $\Big\{ \varnothing, \{0\}, \{0, 1\} \Big\}$. Prove that $\mathbb{S}$ is not metrizable.

Observe that, among the list of three open sets in $\mathbb{S}$, there is no open set that contains the point 1 but not the point 0. If $\mathbb{S}$ were metrizable, this would contradict the conclusion of part (a). Hence, $\mathbb{S}$ is not metrizable.

4. (5 points) Let (X, d) be a metric space, and let $a_\infty \in X$. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in X with the property that every subsequence of $(a_n)_{n\in\mathbb{N}}$ has a subsequence that converges to a_∞. Prove that $(a_n)_{n\in\mathbb{N}}$ converges to a_∞.

We will prove the contrapositive: We will show that if $(a_n)_{n\in\mathbb{N}}$ does not converge to a_∞, then we can construct a subsequence $(a_{n_i})_{i\in\mathbb{N}}$ with the property that none of its (sub)subsequences converges to a_∞.

Recall that a sequence $(a_n)_{n\in\mathbb{N}}$ converges to a_∞ if, for every $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that $a_n \in B_\epsilon(a_\infty)$ for all $n > N$. This means that the sequence $(a_n)_{n\in\mathbb{N}}$ *fails* to converge to a_∞ if for some $\epsilon > 0$, for every $N \in \mathbb{N}$ there is some $n > N$ so that $a_n \notin B_\epsilon(a_\infty)$.

Assume that $(a_n)_{n\in\mathbb{N}}$ does not converge, and choose such a value of $\epsilon > 0$. It follows that there is some $n_1 \in \mathbb{N}$ so that $a_{n_1} \notin B_\epsilon(a_\infty)$. But then, taking $N_1 = n_1$, there is necessarily some $n_2 > n_1$ so that $a_{n_2} \notin B_\epsilon(a_\infty)$. And, again, taking $N_2 = n_2$, there is some $n_3 > n_2$ such that $a_{n_3} \notin B_\epsilon(a_\infty)$. Continuing with this procedure, by induction, we obtain a subsequence $(a_{n_i})_{i\in\mathbb{N}}$ that has no terms contained in $B_\epsilon(a_\infty)$.

We will show that no subsequence of $(a_{n_i})_{i\in\mathbb{N}}$ converges to a_∞. Let $(a_{n_{i_j}})_{j\in\mathbb{N}}$ be any (sub)subsequence of this subsequence. Consider the value $\epsilon > 0$ as defined above. If $(a_{n_{i_j}})_{j\in\mathbb{N}}$ converged to a_∞, then $a_{n_{i_j}}$ must be contained in $B_\epsilon(a_\infty)$ for infinitely many values of j. However, the terms $a_{n_{i_j}}$ by construction are not contained in $B_\epsilon(a_\infty)$ for *any* values of j. Thus, the subsequence does not converge to a_∞.

We conclude that a sequence $(a_n)_{n\in\mathbb{N}}$ must converge to a_∞ if it has the property that each of its subsequences has a (sub)subsequence converging to a_∞.