Name: ____________________ Score (Out of 4 points):

Definition (proper maps). Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A map $f : X \to Y$ is called a *proper map* if $f^{-1}(K) \subseteq X$ is compact for every compact set $K \subseteq Y$.

1. (2 points) Consider the set $\mathbb{R}$ with the Euclidean metric. Show that the constant map

$$f : \mathbb{R} \to \mathbb{R}$$
$$f(x) = 0$$

is continuous but not proper.

Solution: Let $A \subseteq \mathbb{R}$ be any subset. Then

$$f^{-1}(A) = \begin{cases} \varnothing, & 0 \notin A \\ \mathbb{R}, & 0 \in A. \end{cases}$$

In particular, if A is an open set, then its preimage $f^{-1}(A)$ is either $\varnothing$ or $\mathbb{R}$ and so is necessarily open. This shows that f is continuous (a fact we proved on Worksheet #8).

However, f is not proper. We know that the set $\{0\}$ is finite and therefore (by Worksheet #12) compact. However, its preimage is the set $\mathbb{R}$, which is unbounded and therefore (by Worksheet #12) not compact. We conclude that f is not proper.

2. (2 points) Let $(X, \mathcal{T}_X)$ be a compact topological space, and let $(Y, \mathcal{T}_Y)$ be a Hausdorff topological space. Let $f : X \to Y$ be a continuous map. Show that f is proper.

Solution: Let K be a compact subset of Y. We wish to show that $f^{-1}(K)$ is a compact subset of X.

By Worksheet #12, since Y is Hausdorff, the compact subset K must be closed. Thus, since f is continuous, $f^{-1}(K)$ must be closed. But by Worksheet #12, closed subsets of compact spaces are compact. Thus $f^{-1}(K)$ is compact, and we conclude that f is proper.