Name: ______________________________ Score (Out of 8 points):

1. (a) (3 points) Let X be a set. State the definition of a *metric* on X.

A *metric* on X is a function

$$d : X \times X \longrightarrow \mathbb{R}$$

satisfying the following conditions.

(M1) **(Positivity).** $d(x, y) \geq 0$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$.

(M2) **(Symmetry).** $d(x, y) = d(y, x)$ for all $x, y \in X$.

(M3) **(Triangle inequality).** $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$.

(b) (3 points) Define a function

$$d_\infty : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

as follows. For points $\overline{x} = (x_1, \ldots, x_n)$ and $\overline{y} = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, let

$$d_\infty(\overline{x}, \overline{y}) = \max_{i=1,\ldots,n} |x_i - y_i|.$$

Prove that d_∞ satisfies the triangle inequality.

Let $\overline{x} = (x_1, \ldots, x_n)$, $\overline{y} = (y_1, \ldots, y_n)$, and $\overline{z} = (z_1, \ldots, z_n)$ be points in $\mathbb{R}^n$. Our goal is to show that

$$d(\overline{x}, \overline{z}) \leq d(\overline{x}, \overline{y}) + d(\overline{y}, \overline{z}).$$

Observe that, for any fixed $j \in \{1, 2, \ldots, n\}$,

$$|x_j - y_j| \leq \max_{i=1,\ldots,n} |x_i - y_i| = d(\overline{x}, \overline{y}).$$

Moreover, from our proof that the Euclidean metric on $\mathbb{R}$ satisfies the triangle inequality, we know that

$$|a - c| \leq |a - b| + |b - c| \qquad \text{for all } a, b, c \in \mathbb{R}.$$

Hence, for any fixed j, we find

$$\begin{aligned}|x_j - z_j| &\leq |x_j - y_j| + |y_j - z_j| \\ &\leq d(\overline{x}, \overline{y}) + d(\overline{y}, \overline{z}).\end{aligned}$$

Since $|x_j - z_j| \leq d(\overline{x}, \overline{y}) + d(\overline{y}, \overline{z})$ for every $j \in \{1, 2, \ldots, n\}$, it follows that

$$\begin{aligned}d(\overline{x}, \overline{z}) &= \max_{i=1,\ldots,n} |x_i - z_i| \\ &\leq d(\overline{x}, \overline{y}) + d(\overline{y}, \overline{z})\end{aligned}$$

which concludes the proof.

(c) (2 points) In fact, d_∞ defines a metric on $\mathbb{R}^n$. Draw and shade the open ball $B_2(0,0)$ of radius 2 about the origin $(0,0)$ in the metric space $(\mathbb{R}^2, d_\infty)$.

By definition, this ball is all points $(x, y) \in \mathbb{R}^2$ with

$$2 > d((x, y), (0, 0)) = \max(|x - 0|, |y - 0|) = \max(|x|, |y|).$$

The "boundary" of the ball, the points that satisfy

$$2 = d((x, y), (0, 0)) = \max(|x - 0|, |y - 0|) = \max(|x|, |y|),$$

will be points of the form $(2, y)$ or $(-2, y)$ with $-2 \leq y \leq 2$, or $(x, 2)$ or $(x, -2)$ with $-2 \leq x \leq 2$.