Name: ______________________ Score (Out of 4 points):

1. (4 points) Let X be a set and let $d_X : X \times X \to \mathbb{R}$ be the *discrete* metric on X, that is, let

$$d_X(x, x') = \begin{cases} 0, & x = x' \\ 1, & x \neq x' \end{cases} \qquad \text{for all } x, x' \in X.$$

Let (Y, d_Y) be any metric space. Prove that any function $f : X \to Y$ is continuous.

We first prove some preliminary results:

Lemma 1. Let x be any point in X. Then the set $\{x\}$ containing the single set $x \in X$ is open.

Proof. To prove that the set $S = \{x\}$ is open, we must show that x is an interior point of S. But consider the ball $B_{\frac{1}{2}}(x)$ of radius $r = \frac{1}{2}$ about x. Since every other point in X is distance 1 away from x, $B_{\frac{1}{2}}(x) = \{x\} \subseteq S$. Hence x is an interior point of S, and S is open.

Lemma 2. Let $S \subseteq X$ be any subset of X. Then S is open.

Proof. If $S = \varnothing$, then S is open by definition (vacuously, all of its points are interior points). Otherwise, we can express S as a union of sets containing single points,

$$S = \bigcup_{s \in S} \{s\}.$$

But we proved on Worksheet 2 that a union of open sets is open, hence it follows from Lemma 1 that the set S is open.

We are now prepared to prove the result. Recall from Worksheet 3 that the function f is continuous if and only if, for any open set $U \subseteq Y$, the preimage $f^{-1}(U)$ is open in X. Let $U \subseteq Y$ be an arbitrary open set. Lemma 2 shows that **every** subset of X is open, and so in particular $f^{-1}(U)$ will be open. We conclude that f is continuous.