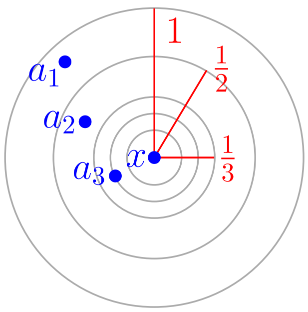
Name: ______________________________ Score (Out of 4 points):

1. (4 points) Let (X, d) be a metric space and let $A \subseteq X$ be a subset. Suppose that A has the property that, given any convergent sequence $(a_n)_{n\in\mathbb{N}}$ of points in A, its limit a_∞ is contained in A. Prove that A is closed.

Solution: We will prove the contrapositive: if A is not closed, then there exists a sequence of points in A that converge to a point in $X \setminus A$.

So suppose that A is not closed. By definition, there is therefore some point x in the complement $X \setminus A$ that is not an interior point of $X \setminus A$. This means that for any choice of $r > 0$, the ball $B_r(x)$ contains at least one point of A.

Construct a sequence as follows. Let a_1 be a point of A in the ball $B_1(x)$. Let a_2 be a point of A in the ball $B_{\frac{1}{2}}(x)$. In general, for $n \in \mathbb{N}$, let a_n be a point of A in the ball $B_{\frac{1}{n}}(x)$.



The resultant sequence $(a_n)_{n\in\mathbb{N}}$ satisfies $\{a_n\}_{n\in\mathbb{N}} \subseteq A$ by construction. We will prove that it converges to $x \notin A$. Let $\epsilon > 0$. Choose N large enough so that $\frac{1}{N} < \epsilon$. Then for all $n \geq N$, we find that $\frac{1}{n} \leq \frac{1}{N} < \epsilon$, and

$$a_n \in B_{\frac{1}{n}}(x) \subseteq B_\epsilon(x).$$

Thus $\lim_{n\to\infty} a_n = x$, which concludes the proof.