Name: ______________________ Score (Out of 8 points):

1. (4 points) Let (X, d) be a metric space, and $A \subseteq X$ a subset. Prove that $\overline{X \setminus A} = X \setminus \mathring{A}$.

Solution: To prove this equality, we will show that $\overline{X \setminus A} \subseteq X \setminus \mathring{A}$ and that $\overline{X \setminus A} \supseteq X \setminus \mathring{A}$.

First suppose $x \subseteq \overline{X \setminus A}$. This means that every neighbourhood U of x contains a point in $X \setminus A$. In other words, if U is a neighbourhood of x, then U contains at least one point not in A. In particular, every open ball $B_r(x)$ is a neighbourhood of x and therefore $B_r(x) \not\subseteq A$ for any $r > 0$. We conclude that x is not an interior point of A, hence $x \in X \setminus \mathring{A}$.

Now suppose that $x \in X \setminus \mathring{A}$. Let U be a neighbourhood of x. Since U is open by definition, there is some $r > 0$ such that $B_r(x) \subseteq U$. Because $x \notin \mathring{A}$ by assumption, x is not an interior point of A, and so no open ball centred on x can be contained in A. In particular, the ball $B_r(x)$ must contain a point $y \in X \setminus A$. But then $y \in B_r(x) \subseteq U$. We conclude that any neighbourhood U of x contains a point $y \in X \setminus A$, and hence that $x \in \overline{X \setminus A}$.

2. (4 points) Consider the the real numbers $\mathbb{R}$ with the Euclidean metric. For each of the following subsets $A \subseteq \mathbb{R}$, state the interior $\mathring{A}$ and the closure $\overline{A}$. No justification necessary.

$A = [-1, 1)$ $\mathring{A} = (-1, 1)$ $\overline{A} = [-1, 1]$

$A = \mathbb{N} = \{1, 2, 3, \ldots\}$ $\mathring{A} = \varnothing$ $\overline{A} = \mathbb{N} = \{1, 2, 3, \ldots\}$

$A = \left\{ \frac{1}{2^n} \;\middle|\; n \in \mathbb{N} \right\}$ $\mathring{A} = \varnothing$ $\overline{A} = \left\{ \frac{1}{2^n} \;\middle|\; n \in \mathbb{N} \right\} \cup \{0\}$

$A = \varnothing$ $\mathring{A} = \varnothing$ $\overline{A} = \varnothing$