Name: ______________________________ Score (Out of 6 points):

1. (3 points) Suppose that $(X, d)$ is a metric space, and that $X$ is a **finite** set. Prove that $X$ is sequentially compact.

   Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of points in $X$. To show that $X$ is sequentially compact, by definition, we must show that $(a_n)_{n\in\mathbb{N}}$ has a convergent subsequence.

   Since $X$ is finite, there must be some element $x \in X$ such that $a_n = x$ for infinitely many values of $n$. Hence, there must be some subsequence $(a_{n_i})_{i\in\mathbb{N}}$ such that $a_{n_i} = x$ for all $i \in \mathbb{N}$.

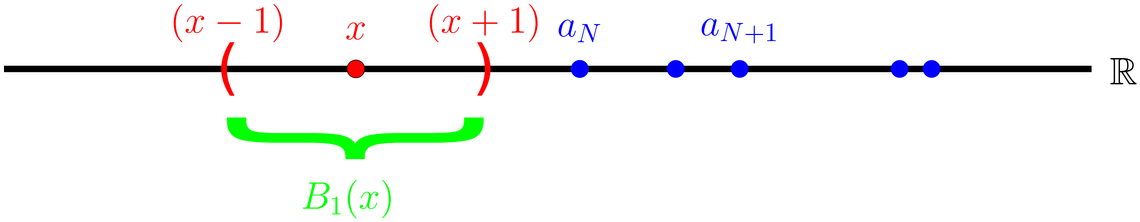
   This subsequence converges to $x$: for any $\epsilon > 0$, $a_{n_i} = x$ is contained in $B_\epsilon(x)$ for all $i \geq 1$.

2. (3 points) Recall that a sequence of real numbers $(a_n)_{n\in\mathbb{N}}$ is said to *diverge to infinity* if for every $R \in \mathbb{R}$, there is some $N \in \mathbb{N}$ so that $a_n \geq R$ for all $n \geq N$.

   Suppose that $(a_n)_{n\in\mathbb{N}}$ is a sequence of real numbers that diverges to infinity. Show that no subsequence of $(a_n)_{n\in\mathbb{N}}$ converges.

   Let $(a_n)_{n\in\mathbb{N}}$ is a sequence of real numbers that diverges to infinity, and let $(a_{n_i})_{i\in\mathbb{N}}$ be a subsequence. We will prove that this subsequence does not converge to any real number $x$.

   So fix $x \in \mathbb{R}$, and let $\epsilon = 1$. Choose $R = x + 1$. Then, by assumption, there is an $N \in \mathbb{N}$ so that $a_n \geq R = x + 1$ for every $n \geq N$.




   In particular, whenever $i \geq N$, then $n_i \geq i \geq N$, and $a_{n_i} \geq x + 1$. This means that $a_{n_i} \notin B_1(x) = (x - 1, x + 1)$ whenever $i \geq N$. We conclude that the subsequence does not converge to $x$.

   Since $x$ was an arbitrary real number, and $(a_{n_i})_{i\in\mathbb{N}}$ an arbitrary subsequence, we conclude that no subsequence of $(a_n)_{n\in\mathbb{N}}$ converges.