Name: ______________________ Score (Out of 5 points):

1. (1 point) Let $X = \{a, b, c, d\}$. Let $\mathcal{T}$ be the topology on X

$$\mathcal{T} = \{\varnothing, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}, X\}.$$

(You do not need to verify that this is a topology). What is the subspace topology $\mathcal{T}_S$ induced on the subset $S = \{a, b, d\} \subseteq X$?

Solution: By definition, the elements of $\mathcal{T}_S$ are the intersections of S and elements of $\mathcal{T}$. These are the sets:

$$\mathcal{T}_S = \{\varnothing, \{b\}, \{a, b\}, \{b, d\}, S\}$$

2. Suppose that $(X, \mathcal{T})$ is a topological space, and that $S \subseteq X$ a subset. Define a map

$$i : S \to X$$
$$i(s) = s.$$

This map is called the *inclusion* of S into X.

(a) (2 points) Show that this map is continuous with respect to the topology $\mathcal{T}$ on X and the subspace topology $\mathcal{T}_S$ on S.

Solution: To show that the map i is continuous, we must show that $i^{-1}(U)$ is open for every open set $U \subseteq X$. So let $U \subseteq X$ be open. Then

$$i^{-1}(U) = \{ s \in S \mid i(s) \in U \} = \{ s \in S \mid s \in U \} = U \cap S.$$

But $U \cap S$ is open in S by definition of the subset topology, so we conclude that i is continuous.

(b) (2 points) Suppose that S is an **open** set in the topological space X. Show that the map i is an open map. (Recall that this means that $i(V) \subseteq X$ is open for every open set $V \subseteq S$.)

Solution: To prove that i is open, we must show that $i(V) \subseteq X$ is open for every open set $V \subseteq S$. By definition of i,

$$i(V) = \{ i(v) \mid v \in V \} = \{ v \mid v \in V \} = V.$$

By definition of the subspace topology $\mathcal{T}_S$, a set $V \subseteq S$ is open precisely when $V = U \cap S$ for some open set $U \subseteq X$. But, then $U \in \mathcal{T}$, and $S \in \mathcal{T}$ by assumption, and so by definition of a topology their intersection $S \cap U \in \mathcal{T}$ must also be open in X. Hence $i(V) = V = S \cap U$ is an open subset of X, as claimed.