Name: ______________________________ Score (Out of 5 points):

1. (2 points) Let $X = \{0, 1, 2\}$ be the topological space with the topology

$$\mathcal{T} = \{\varnothing, \{0\}, \{0,1\}, \{0,1,2\}\}.$$

Let $(a_n)_{n\in\mathbb{N}}$ be the constant sequence $1, 1, 1, 1, \cdots$. List all elements $x \in X$ such that $(a_n)_{n\in\mathbb{N}}$ converges to x. **No justification necessary**.

Solution: 1, 2.

By definition, $(a_n)_{n\in\mathbb{N}}$ converges to x if, for every open neighbourhood U of x, there is some $N \in \mathbb{N}$ so that $a_n \in U$ for all $n \geq N$. Since the sequence in question is the constant sequence $a_n = 1$, in this case, $(a_n)_{n\in\mathbb{N}}$ converges to x if and only if every open neighbourhood U of x contains the point 1.

We can consider each of the points 0, 1, and 2 of X. The sequence does **not** converge to 0, since the open neighbourhood $\{0\}$ of 0 does not contain the point 1. The sequence **does** converge to 1, since (by definition) every open neighbourhood of 1 contains 1. The sequence also converges to the point 2, since the only open neighbourhood of 2 is the set $\{0, 1, 2\}$, which also contains the point 1.

2. (3 points) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and let $f : X \to Y$ be a continuous map. Suppose that $(a_n)_{n\in\mathbb{N}}$ is a sequence in X that converges to a point $a_\infty \in X$. Show that the sequence $\Big(f(a_n)\Big)_{n\in\mathbb{N}}$ in Y converges to the point $f(a_\infty) \in Y$.

Solution: Let $U \subseteq Y$ be any open neighbourhood of $f(a_\infty)$. To show that

$$\lim_{n\to\infty} f(a_n) = f(a_\infty),$$

we must find some $N \in \mathbb{N}$ such that $f(a_n) \in U$ for all $n \geq N$.

Consider the preimage $f^{-1}(U)$. Because U is open and f is continuous, we know that $f^{-1}(U)$ is open. Moreover, since $f(a_\infty) \in U$, by definition we know that $a_\infty \in f^{-1}(U)$. Thus $f^{-1}(U)$ is an open neighbourhood of a_∞.

The assumption that $(a_n)_{n\in\mathbb{N}}$ converges to a_∞ then implies that there exists $N \in \mathbb{N}$ so that $a_n \in f^{-1}(U)$ for all $n \geq N$. But this means that $f(a_n) \in U$ for all $n \geq N$. We conclude that $(f(a_n))_{n\in\mathbb{N}}$ converges to to $f(a_\infty)$, as claimed.