Name: ______________________ Score (Out of 6 points):

1. (3 points) Let $X = \{a, b, c, d\}$ be the topological space with the topology

$$\mathcal{T} = \big\{\varnothing, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\big\}.$$

Which of the following sets are bases for this topology? Circle all that apply.
No further justification necessary.

(i) $\mathcal{B} = \{\varnothing, X\}$
These sets do not generate $\mathcal{T}$, for example, $\{a\}$ cannot be written as a union of elements of $\mathcal{B}$.

(ii) $\mathcal{B} = \{\{a\}, \{b\}, \{c\}, \{d\}\}$
Basis elements must be open sets, and here $\{c\}$, $\{d\} \notin \mathcal{T}$.

(iii) $\mathcal{B} = \mathcal{T}$
$\mathcal{T}$ is always generated by $\mathcal{T}$.

(iv) $\mathcal{B} = \{\{a\}, \{b\}, \{b, c\}, \{a, d\}\}$
These elements do form a basis: every set is open, and they generate $\mathcal{T}$. For example, $\{a, b, c\}$ is the union of basis elements $\{a\} \cup \{b, c\}$.

(v) $\mathcal{B} = \{\{a\}, \{b\}, \{a, b, c\}, \{a, b, d\}\}$
These sets do not generate $\mathcal{T}$, for example, $\{a, d\}$ cannot be written as a union of elements of $\mathcal{B}$.

(vi) $\mathcal{B} = \{\{a, b\}, \{b, c\}, \{a, d\}\}$
These sets do not generate $\mathcal{T}$, for example, $\{a\}$ cannot be written as a union of elements of $\mathcal{B}$.

2. (3 points) Let $(X, \mathcal{T})$ be a topological space, and suppose that $\mathcal{B}$ is a basis for this topology. Let $A \subseteq X$. Show that a point $x \in X$ is contained in $\overline{A}$ if and only if every basis element $B \in \mathcal{B}$ containing x contains a point of A.

First, suppose that $x \in \overline{A}$. By definition, this means that every open neighbourhood U of x will contain a point of A. In particular, any open neighbourhood of x contained in $\mathcal{B}$ must contain a point of A.

Conversely, suppose that $x \in X$ has the property that every element of $B \in \mathcal{B}$ containing x also contains a point of A. Let U be any open neighbourhood of x; we must show that U contains a point of A. Because $\mathcal{B}$ is a basis for the topology, we can write U as a union of basis elements $U = \cup_{i \in I} B_i$, with $B_i \in \mathcal{B}$. Since $x \in U$, there must be some index i such that $x \in B_i$. But then by assumption B_i contains some element $a \in A$. So $a \in B_i \subseteq U$, and we conclude that U contains a point of A, as desired.