Name: ______________________ Score (Out of 5 points):

1. (2 points) Let $X = \{a, b, c, d\}$ be a topological space with the topology

$$\mathcal{T} = \Big\{\varnothing, \{a\}, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}\Big\}.$$

Write down a formula for a continuous path in X from a to d. **No justification necessary.**

We seek a function $\gamma : [0, 1] \to X$ with the property that

$$\gamma(0) = a, \qquad \gamma(1) = d, \qquad \gamma^{-1}(\{a\}), \gamma^{-1}(\{a, b\}), \gamma^{-1}(\{a, b, c\}) \text{ are open.}$$

One such function is the following:

$$\gamma : [0,1] \longrightarrow X$$
$$\gamma(t) = \begin{cases} a, & t \in [0, \frac{1}{8}), \\ b, & t \in [\frac{1}{8}, \frac{1}{4}), \\ c, & t \in [\frac{1}{4}, \frac{1}{2}), \\ d, & t \in [\frac{1}{2}, 1] \end{cases}$$

Then

$$\gamma^{-1}(\varnothing) = \varnothing, \qquad \gamma^{-1}(\{a\}) = \left[0, \frac{1}{8}\right), \qquad \gamma^{-1}(\{a, b\}) = \left[0, \frac{1}{4}\right),$$
$$\gamma^{-1}(\{a, b, c\}) = \left[0, \frac{1}{2}\right), \qquad \gamma^{-1}(\{a, b, c, d\}) = [0, 1]$$

are all open sets in $[0, 1]$, so the function is continuous as desired.

2. (3 points) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and $f : X \to Y$ a continuous function. Show that, if X is path-connected, then $f(X)$ is path-connected.

Let y_0 and y_1 be any two points in $f(X)$. To show that $f(X)$ is path-connected, we must show that there exists a path $\gamma : [0, 1] \to f(X)$ such that $\gamma(0) = y_0$ and $\gamma(1) = y_1$.

Since y_0 and y_1 are in the image of f by assumption, there must exist points x_0 and x_1 in X such that $f(x_0) = y_0$ and $f(x_1) = y_1$. Since X is path-connected, there exists a path $\widetilde{\gamma} : [0, 1] \to X$ with $\widetilde{\gamma}(0) = x_0$ and $\widetilde{\gamma}(1) = x_1$.

Let $\gamma = f \circ \widetilde{\gamma}$. The map γ must be continuous, since both f and $\widetilde{\gamma}$ are continuous. Moreover,

$$\gamma(0) = f \circ \widetilde{\gamma}(0) = f(x_0) = y_0 \qquad \text{and} \qquad \gamma(1) = f \circ \widetilde{\gamma}(1) = f(x_1) = y_1.$$

Thus γ is the desired path, and we conclude that $f(X)$ is path-connected.