Review Problems

1. (a) Let $(X, \mathcal{T}_X)$ be a Hausdorff topological space, and let $x_1, \ldots, x_n$ be a finite collection of points in X. Show that there are open sets $U_1, \ldots, U_n$ such that $x_i \in U_i$, and which are pairwise disjoint (this means $U_i \cap U_j = \varnothing$ for all $i \neq j$).
   (b) Let X be a **finite** set with a topology $\mathcal{T}_X$. Prove that if $\mathcal{T}_X$ is Hausdorff, then $\mathcal{T}_X$ is the discrete topology.
2. Let X be a set. Show that $\mathcal{B} = \left\{ \{x\} \;\middle|\; x \in X \right\}$ is a basis for the discrete topology $\mathcal{T}$ on X.
3. Let $(X, \mathcal{T})$ be a topological space. Show that a set $\mathcal{B}$ is a basis for $\mathcal{T}$ if and only if it satisfies the following property: For every open set $U \in \mathcal{T}$, and every $x \in U$, there exists some element $B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq U$.
4. Let $(X, \mathcal{T}_X)$ be a topological space, and suppose that the topology $\mathcal{T}_X$ is induced by a metric d on X. Let $S \subseteq X$. Show that the subspace topology $\mathcal{T}_S$ is equal to the topology on S induced by restricting the metric d to S.
5. Let $(X, \mathcal{T})$ be a topological space with basis $\mathcal{B}$. Suppose that $S \subseteq X$ is a subset, and consider S as a topological space with the induced subspace topology $\mathcal{T}_S$. Prove that $\mathcal{B}_S = \{B \cap S \mid B \in \mathcal{B}\}$ is a basis for $\mathcal{T}_S$.
6. Let $(X, \mathcal{T})$ be a topological space with basis $\mathcal{B}$. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of points in X. Show that this sequence converges to a point a_∞ if and only if, for every set $B \in \mathcal{B}$ containing a_∞, there is some $N \in \mathbb{N}$ so that $a_n \in B$ for all $n \geq N$.
7. Let $(X, \mathcal{T})$ be a topological space with basis $\mathcal{B}$. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and let $\mathcal{B}$ be a basis for $\mathcal{T}_X$. Prove that a map $f : X \to Y$ is open if and only if $f(B)$ is open for every $B \in \mathcal{B}$.
8. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Let $\mathcal{B}_X$ be a basis for $\mathcal{T}_X$, and let $\mathcal{B}_Y$ be a basis for $\mathcal{T}_Y$. Show that the set
$$\mathcal{B}_{X \times Y} = \{\, B_X \times B_Y \mid B_X \in \mathcal{B}_X, B_Y \in \mathcal{B}_Y \,\}$$
is a basis for the product topology $\mathcal{T}_{X \times Y}$ on $X \times Y$.
9. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be (nonempty) topological spaces. Let $\mathcal{T}_{X \times Y}$ be the product topology on $X \times Y$. Suppose that the topologies $\mathcal{T}_X$ and $\mathcal{T}_Y$ are induced by metrics d_X on X and d_Y on Y. Show that the product topology is induced by the metric
$$\begin{aligned} d_{X \times Y} :\ & (X \times Y) \times (X \times Y) \longrightarrow \mathbb{R} \\ d_{X \times Y}\Big((x_1, y_1), (x_2, y_2)\Big) &= \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2}. \end{aligned}$$
10. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be (nonempty) topological spaces. Let $\mathcal{T}_{X \times Y}$ be the product topology on $X \times Y$.
    (a) Define the projection map π_X by
    $$\begin{aligned} \pi_X : X \times Y &\to X \\ \pi_X(x, y) &= x \end{aligned}$$
    Show that the projection map π_X is continuous.
    (b) Show that the projection map π_X is open.
    (c) Fix a point $y \in Y$. Consider the map
    $$\begin{aligned} f_X : X &\to X \times Y \\ f_X(x) &= (x, y) \end{aligned}$$
    Is the map f_X necessarily continuous?
    (d) Is the map f_X necessarily open?

Bonus Problems (Optional): Coarser and finer topologies

10. Consider the following definitions.

    Definition (Coarser topology; finer topology). Let X be a set. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on X. If $\mathcal{T}_1 \subseteq \mathcal{T}_2$, then the topology $\mathcal{T}_1$ is said to be *coarser* than $\mathcal{T}_2$, and the topology $\mathcal{T}_2$ is said to be *finer* than the topology $\mathcal{T}_1$.

    (a) Let X be a set. Show that the indiscrete topology on X is coarser than any other topology on X.

    (b) Let X be a set. Show that the discrete topology on X is finer than any other topology on X.

11. Let X be a set. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on X. Show that the following statements are equivalent.

    - $\mathcal{T}_1 \subseteq \mathcal{T}_2$
    - The identity map $I : (X, \mathcal{T}_2) \to (X, \mathcal{T}_1)$ is continuous
    - The identity map $I : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is open

12. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Let $f : X \to Y$ be a function.

    (a) Suppose that f is continuous. Show that f will still be continuous if we replace $\mathcal{T}_X$ by any finer topology on X, or if we replace $\mathcal{T}_Y$ with any coarser topology on Y.

    (b) Suppose that f is an open map. Show that f will still be open if we replace $\mathcal{T}_X$ by any coarser topology on X, or if we replace $\mathcal{T}_Y$ by any finer topology on Y.

13. Let $(X, \mathcal{T}_X)$ be a topological space, and let $S \subseteq X$. Show that the subspace topology $\mathcal{T}_S$ is the coarsest topology that makes the inclusion map $i : S \to X$ continuous. (In other words, show that if $\mathcal{T}$ is any topology on S such that $i : (S, \mathcal{T}) \to (X, \mathcal{T}_X)$ is continuous, then $\mathcal{T}_S \subseteq \mathcal{T}$.)

14. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Show that the product topology $\mathcal{T}_{X \times Y}$ on $X \times Y$ is the coarsest topology on $X \times Y$ that makes the projection maps π_X and π_Y continuous.

Bonus Problems (Optional): The p–adic topology

15. Let p be a prime number. For $q \in \mathbb{Q}$, define the *p-adic absolute value*

$$|q|_p = \begin{cases} 0, & \text{if } q = 0 \\ p^{-r}, & \text{if } q = p^r \frac{a}{b}, \quad a, b \in \mathbb{Z}, \text{ neither } a \text{ nor } b \text{ are divisible by } p. \end{cases}$$

    (a) Prove that, for $x, y \in \mathbb{Q}$, $|x + y|_p \leq \max\{|x|_p, |y|_p\}$.

    (b) Under what conditions on x and y will $|x + y|_p = \max\{|x|_p, |y|_p\}$?

    (c) Prove that the function $d_p(x, y) = |x - y|_p$ defines a metric on $\mathbb{Q}$ (called the *p-adic metric*).

    (d) If p were a composite number, would this function d_p still be a metric?

    (e) Suppose the definition of $|q|_p$ were changed by replacing p^{-r} with p^r. Would d_p still define a metric?

    (f) Let $p = 3$. Describe the balls $B_9(0)$, $B_1(0)$, and $B_{\frac{1}{3}}(0)$ in the 3-adic metric.