Review Problems

1. Suppose that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces and that $f : X \to Y$ is a continuous function. Let $A \subseteq X$ be a subset of X, endowed with the subset topology $\mathcal{T}_A$. Show that the restriction $f|_A$ of f to A is continuous, viewed as a map from $(A, \mathcal{T}_A)$ to $(Y, \mathcal{T}_Y)$.

2. Suppose that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces and that $f : X \to Y$ is a function.
   (a) Suppose that A and B are **open** subsets of X, such that $A \cup B = X$ and $f|_A$ and $f|_B$ are both continuous (with respect to the subspace topologies). Prove that f is continuous.
   (b) Now suppose that A and B are **arbitrary** subsets of X, such that $A \cup B = X$ and $f|_A$ and $f|_B$ are both continuous. Either prove that f is continuous, or show by counterexample that f need not be continuous.

3. Let $(X, \mathcal{T})$ be a topological space, and $A \subseteq X$ a subset.
   (a) Prove or find a counterexample: if A is connected, then X is connected.
   (b) Prove or find a counterexample: if X is connected, then A is connected.

4. Let $(X, \mathcal{T})$ be a Hausdorff topological space, and let $A \subseteq X$ be a finite subset containing at least two elements. Show that A is disconnected.

5. Let $X = \{a, b, c, d\}$. Let $\mathcal{T}$ be the topology on X
$$\mathcal{T} = \{\varnothing, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}, X\}.$$
Determine which of following subsets are connected, and which are disconnected.
   (a) X (b) $\{b, c\}$ (c) $\{a, b\}$ (d) $\{a, b, c\}$

6. Let $(X, \mathcal{T})$ be a topological space. Suppose that A and B are connected subsets of X with $A \cap B \neq \varnothing$.
   (a) Show that $A \cup B$ is connected.
   (b) Show by example that $A \cap B$ may be disconnected. *Hint:* Consider subsets of $\mathbb{R}^2$.

7. **(Challenge Problem).**

   Definition (Sequential continuity). Let $f : X \to Y$ be a continuous function of topological spaces. Then f is called *sequentially continuous* if for any convergent sequence $(x_n)_{n\in\mathbb{N}}$ in X, the sequence $(f(x_n))_{n\in\mathbb{N}}$ converges and
$$\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right).$$

   You proved on Quiz #7 that continuous functions are sequentially continuous. You proved moreover on Worksheet #4 that, if X and Y are metric spaces, a function is continuous if and only if it is sequentially continuous. In this problem, we will see that the converse does **not** hold for general topological spaces.
   (a) Recall that a set S is *countable* if there exists an injective function $S \to \mathbb{N}$. Such sets are either finite or countably infinite. Define a collection of subsets of $\mathbb{R}$
$$\mathcal{T}_{cc} = \{\varnothing\} \cup \{\ U \subseteq \mathbb{R} \mid\ \mathbb{R} \setminus U \text{ is countable }\}.$$
   Show that $\mathcal{T}_{cc}$ is a topology on $\mathbb{R}$, called the *co-countable topology.*
   (b) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $(\mathbb{R}, \mathcal{T}_{cc})$. Show that $(a_n)_{n\in\mathbb{N}}$ converges if and only if it is *eventually constant.* This means there is some $N \in \mathbb{N}$ and $x \in \mathbb{R}$ so that $a_n = x$ for all $n \geq N$.
   (c) Let $\mathcal{T}_{dsc}$ denote the discrete topology on $\mathbb{R}$. Let $I : \mathbb{R} \to \mathbb{R}$ be the identity map. Show that the map of topological spaces
$$I : (\mathbb{R}, \mathcal{T}_{cc}) \to (\mathbb{R}, \mathcal{T}_{dsc})$$
   is **not** continuous.
   (d) Show that the map of topological spaces $I : (\mathbb{R}, \mathcal{T}_{cc}) \to (\mathbb{R}, \mathcal{T}_{dsc})$ is sequentially continuous.