Review Problems

1. Let $(X, \mathcal{T})$ be a topological space. On Worksheet #9 Problem 1(b), you proved that, if X is Hausdorff, then all singleton sets $\{x\}$ are closed. Prove or give a counterexample: If all singleton sets $\{x\}$ are closed, then X is Hausdorff.

2. Consider the real numbers $\mathbb{R}$ with the cofinite topology $\mathcal{T}_{cf}$.
   (a) Consider the constant sequence $1\ 1\ 1\ 1\ 1\ 1\ \cdots$. To which $x \in \mathbb{R}$ (if any) does this sequence converge?
   (b) Consider the alternating sequence $0\ 1\ 0\ 1\ 0\ 1\ \cdots$. To which real numbers x (if any) does this sequence converge?
   (c) Consider the sequence of natural numbers $1\ 2\ 3\ 4\ 5\ 6\ \cdots$. To which real numbers x (if any) does this sequence converge?
   (d) In general, let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that $\{\ a_n \mid n \in \mathbb{N}\ \}$ is infinite. To which real numbers x (if any) does this sequence converge?
   (e) Now consider a sequence $(a_n)_{n\in\mathbb{N}}$ such that $\{\ a_n \mid n \in \mathbb{N}\ \}$ is is finite. Under what conditions does this sequence converge? To what real numbers does it converge?

3. **(Challenge Problem).** Suppose (X, d) is a metric space, and $A \subseteq X$. We proved that, if $x \in \overline{A}$, then there is some sequence of points $(a_n)_{n\in\mathbb{N}}$ in A that converge to x. In this problem, we will see that this property does **not** hold for general topological spaces.
   (a) Recall that the *co-countable* topology on $\mathbb{R}$ is the topology

   $$\mathcal{T}_{cc} = \{\varnothing\} \cup \{\ U \subseteq \mathbb{R} \mid\ \mathbb{R} \setminus U \text{ is countable }\}.$$

   Let $A \subseteq \mathbb{R}$ be any subset of $\mathbb{R}$. What is the closure of $\mathbb{R}$ if A is (i) countable, or (ii) uncountable?
   (b) Let $A = (0, 1)$, so $\overline{A} = \mathbb{R}$. Show that, for any $x \in \overline{A} \setminus A$, there is **no** sequence of points in A that converge to x.

Bonus Problems: first countable spaces

1. **Definition (First countable spaces).** A topological space $(X, \mathcal{T})$ is called *first countable* if each point $x \in X$ has a *countable neighbourhood basis*. This means, for each $x \in X$, there is a countable collection $\{N_i\}_{i\in\mathbb{N}}$ of neighbourhoods of x with the property that, if N is any neighbourhood of x, then there is some i such that $N_i \subseteq N$.
   (a) Let (X, d) be a metric space. Show that the topology induced by d is first countable.
   (b) Let X be a set with the discrete topology. Show that X is first countable.
   (c) Let X be a set with the indiscrete topology. Show that X is first countable.
   (d) Let X be the real numbers $\mathbb{R}$ with the cofinite topology. Show that X is not first countable.
   (e) Let X be the real numbers $\mathbb{R}$ with the co-countable topology. Show that X is not first countable.
   (f) Let $(X, \mathcal{T})$ be a first countable topological space. Show that any subspace of X (with the subspace topology) is first countable.
   (g) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be first countable topological spaces. Show that their product $X \times Y$ (with the product topology) is first countable.
   (h) Let X be a first countable space, and let $A \subseteq X$. Show that, given any $x \in \overline{A}$, there is some sequence of points in A that converges to x.
   (i) Let $(X, \mathcal{T}_X)$ be a first countable topological space, and let $(Y, \mathcal{T}_Y)$ be any topological space. Show that a function $f : X \to Y$ is continuous if and only if it is sequentially continuous.