Review Problems

1. Let $(X, \mathcal{T})$ be a topological space, and let $S \subseteq X$ be a subset. Suppose that the subspace topology on S is the discrete topology.
   (a) Prove or give a counterexample: S is closed as a subset of X.
   (b) Prove or give a counterexample: The closure of S in X also has the discrete subspace topology.

2. Let $(X, \mathcal{T})$ be a topological space. Show that X is Hausdorff if and only if, for each $x \in X$,
$$\{x\} = \bigcap_{U \text{ a neighbourhood of } x} \overline{U}.$$

3. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and let $f : X \to Y$ be a function. Recall that the *graph* of f is defined to be the subset of $X \times Y$
$$\{ (x, f(x)) \in X \times Y \mid x \in X \}.$$
Suppose that Y is Hausdorff. Show that, if f is continuous, then the graph of f is a closed subset of $X \times Y$ with respect to the subspace topology $\mathcal{T}_{X \times Y}$.

4. Consider $\mathbb{R}$ with the Euclidean metric. Which of the following subsets are connected?
$$\{x \in \mathbb{R} \mid d(x, 1) < 1 \text{ or } d(x, -1) < 1\}$$
$$\{x \in \mathbb{R} \mid d(x, 1) \leq 1 \text{ or } d(x, -1) < 1\}$$
$$\{x \in \mathbb{R} \mid d(x, 1) \leq 1 \text{ or } d(x, -1) \leq 1\}$$

5. **Definition (Convex subsets of $\mathbb{R}^n$).** Let A be a subset of $\mathbb{R}^n$ (with the Euclidean metric). Then A is called *convex* if $tx + (1 - t)y \in A$ for every $x, y \in A$ and any $t \in [0, 1]$.

   Prove that any convex subset of $\mathbb{R}^n$ is connected.

6. Let $(X, \mathcal{T})$ be a topological space. Let $\{ A_n \mid n \in \mathbb{Z} \}$ be a family of connected subspaces of X such that $A_n \cap A_{n+1} \neq \varnothing$ for every n. Prove $\bigcup_{n \in \mathbb{Z}} A_n$ is connected.

7. Let $(X, \mathcal{T})$ be a topological space. Let $\{A_n \mid n \in \mathbb{N}\}$ be a family of connected subspaces in X such that $A_{n+1} \subseteq A_n$ for every $n \in \mathbb{N}$. Is $\bigcap_{n \in \mathbb{N}} A_n$ is necessarily connected?

8. **(Challenge.)** In this problem, we will construct a topological space that is Hausdorff but not metrizable. For any pair of coprime integers $a, b \in \mathbb{N}$ define
$$N_{a,b} = \{a + kb \mid k \in \mathbb{N} \cup \{0\}\} \subseteq \mathbb{N}.$$
Prove the following.
   (a) The family $\mathcal{B} = \{ N_{a,b} \mid a, b \in \mathbb{N}, \gcd(a, b) = 1\}$ is a basis for a topology $\mathcal{T}$ on $\mathbb{N}$.
   (b) The topology $\mathcal{T}$ is Hausdorff.
   (c) Any mutliple of b is contained in the closure of $N_{a,b}$.
   (d) For any pair of nonempty open sets in $A, B \in \mathcal{T}$ we have $\overline{A} \cap \overline{B} \neq \varnothing$.
   (e) $\mathcal{T}$ is not metrizable.