1 Metric Spaces

Definition 1.1. Let X be a set. A *metric* on X is a function

$$d : X \times X \longrightarrow \mathbb{R}$$

satisfying the following conditions.

(M1) **(Positivity).** $d(x, y) \geq 0$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$.

(M2) **(Symmetry).** $d(x, y) = d(y, x)$ for all $x, y \in X$.

(M3) **(Triangle inequality).** $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$.

The value $d(x, y)$ is sometimes called the *distance from x to y*.

A set X endowed with a metric d is called a *metric space*, and is denoted (X, d) (or simply X when the metric is clear from context).

Theorem 1.2. (The Euclidean Metric). *Define*

$$d : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

as follows. For $\overline{x} = (x_1, \ldots, x_n)$ and $\overline{y} = (y_1, \ldots, y_n)$, let

$$\begin{aligned} d(\overline{x}, \overline{y}) &= ||\overline{x} - \overline{y}|| \\ &= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}. \end{aligned}$$

Then d is a metric, called the Euclidean metric, *and makes $(\mathbb{R}^n, d)$ into a metric space.*

Proof. We need to verify that d satisfies the three conditions that define a metric.

Step 1. Verify that d satisfies condition (M1).

Step 2. Verify that d satisfies condition (M2).

Step 3. Explain why, to verify (M3), it's enough to check that

$$(d(\overline{x},\overline{y})+d(\overline{y},\overline{z}))^2 \geq d(\overline{x},\overline{z})^2$$

Step 4. Expand $(d(\overline{x},\overline{y})+d(\overline{y},\overline{z}))^2 = (||\overline{x}-\overline{y}||+||\overline{y}-\overline{z}||)^2$.

Step 5. Expand

$$\begin{aligned} d(\overline{x},\overline{z})^2 &= (\overline{x}-\overline{z})\cdot(\overline{x}-\overline{z}) \\ &= \Big((\overline{x}-\overline{y})+(\overline{y}-\overline{z})\Big)\cdot\Big((\overline{x}-\overline{y})+(\overline{y}-\overline{z})\Big) \end{aligned}$$

Step 6. Conclude that d satisfies (M3).

$\square$

In-class Exercises

1. Determine whether the following functions define metrics on the corresponding sets. Rigorously justify your answers!

   (a) Let $X = \mathbb{R}$. Define

   $$\begin{aligned} d : \mathbb{R} \times \mathbb{R} &\longrightarrow \mathbb{R} \\ d(x, y) &= (x - y)^2. \end{aligned}$$

   (b) Let $X = \mathbb{R}^2$. Define

   $$\begin{aligned} d : \mathbb{R}^2 \times \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ d(\overline{x}, \overline{y}) &= |x_1 - y_1| + |x_2 - y_2|. \end{aligned}$$

   (c) Let X be any set. Define

   $$\begin{aligned} d : X \times X &\longrightarrow \mathbb{R} \\ d(x, y) &= \begin{cases} 0 & x = y \\ 1 & x \neq y. \end{cases} \end{aligned}$$

2. Let (X, d) be a metric space, and let $Y \subseteq X$ be a subset. Show that the restriction $d|_{Y \times Y}$ of d to $Y \times Y \subseteq X \times X$ defines a metric on Y. Conclude that any subset of a metric space inherits a metric space structure.