1 Connected topological spaces

Definition 1.1. (Disconnected spaces; connected spaces.) A topological space $(X, \mathcal{T})$ is *disconnected* if there exist disjoint nonempty open subsets A and B in X such that $X = A \cup B$. Otherwise, X is called *connected*.

A subset A of X is said to be *connected* if it is connected in the subspace topology $(A, \mathcal{T}_A)$. This means ...

In-class Exercises

1. Show that the following topological spaces (with the Euclidean metric) are disconnected.

   (a) $\mathbb{N}$
   (b) $\{\frac{1}{n} \mid n \in \mathbb{N}\}$
   (c) $(0,1) \cup \{5\}$
   (d) $(0,1) \cup (1,3)$
   (e) $\mathbb{Q}$
   (f) $\mathbb{R} \setminus \mathbb{Q}$

2. Prove that a topological space $(X, \mathcal{T})$ is disconnected if and only if there is subset A, with $\varnothing \subsetneq A \subsetneq X$, that is both open and closed.

3. Consider $\{0, 1\}$ as a topological space with the discrete topology. Show that a topological space $(X, \mathcal{T})$ is disconnected if and only if there is a continuous **surjective** function $X \to \{0, 1\}$.

4. (a) Prove that any set X with the indiscrete topology $\mathcal{T}$ is connected.
   (b) Let X be a set with at least two elements, endowed with the discrete topology. Prove that X is disconnected.

5. Suppose that $(X, \mathcal{T}_X)$ is a **connected** topological space, and $(Y, \mathcal{T}_Y)$ is any topological space. Let $f : X \to Y$ be a continuous map. Prove that if X is connected, then $f(X)$ is connected. In other words, the continuous image of a connected space is connected.