1 Connected and path-connected topological spaces

Definition 1.1. (Path-connected spaces.) Consider the interval $[0,1]$ as a topological space with the topology induced by the Euclidean metric. A topological space $(X, \mathcal{T})$ is *path-connected* if, given any two points $x, y \in X$, there exists a continuous function $\gamma : [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$.

In-class Exercises

1. Suppose that $\{A_i\}_{i \in I}$ is a collection of connected subsets of a topological space $(X, \mathcal{T})$. Show that, if the intersection $\bigcap_{i \in I} A_i$ is nonempty, then the union $\bigcup_{i \in I} A_i$ is connected.
2. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces.
   (a) Suppose that $X \times Y$ is connected in the product topology $\mathcal{T}_{X \times Y}$. Prove that X and Y are connected.
   (b) Suppose that X and Y are connected, and suppose that $(a, b) \in X \times Y$. Prove that $\Big(X \times \{b\}\Big) \cup \Big(\{a\} \times Y\Big)$ is a connected subset of the product $X \times Y$ with the product topology $\mathcal{T}_{X \times Y}$.
   (c) Suppose that X and Y are connected. Prove that $X \times Y$ is connected in the product topology $\mathcal{T}_{X \times Y}$.
3. Let X be a (nonempty) topological space with the indiscrete topology. Is X necessarily path-connected?
4. Recall that Sierpiński space is the space $\mathbb{S} = \{0, 1\}$ with the topology $\mathcal{T} = \{\varnothing, \{0\}, \{0, 1\}\}$.
   (a) Is $\mathbb{S}$ connected?
   (b) Is $\mathbb{S}$ path-connected?
5. Let $(X, \mathcal{T})$ be a topological space. Show that, if X is path-connected, then X is connected. *Hint:* You may use the result from Homework #9 that the interval $[0,1]$ is connected.

6. **Bonus (Optional).** Recall that $\mathcal{C}(0,1)$ denotes the set of continuous functions from the closed interval $[0,1]$ to $\mathbb{R}$, and that $\mathcal{C}(0,1)$ is a metric space with metric

$$d_\infty : \mathcal{C}(0,1) \times \mathcal{C}(0,1) \longrightarrow \mathbb{R}$$
$$d(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)|$$

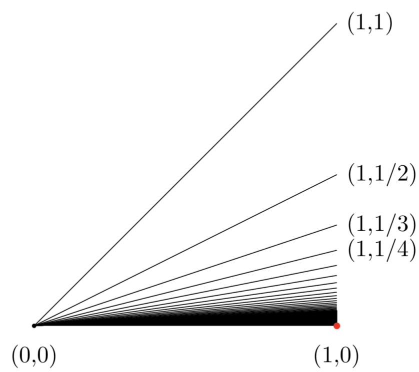
Show that this metric space is path-connected, and therefore connected.

7. **Bonus (Optional).** This problem shows that the converse to Problem 5 fails.

Let X be the following subspace of $\mathbb{R}^2$ (with topology induced by the Euclidean metric)

$$X = \{(1,0)\} \cup \bigcup_{n \in \mathbb{N}} L_n,$$

where L_n is the closed line segment connecting the origin $(0,0)$ to the point $(1, \frac{1}{n})$.



(a) Show that X is connected.
(b) **(Challenge).** Show that X is **not** path-connected.
(c) Would the space be path–connected if we added in the line segment from $(0,0)$ to $(1,0)$?