

1 Compact topological spaces

Definition 1.1. (Open covers; open subcovers.) Let (X, \mathcal{T}) be a topological space. A collection $\{U_i\}_{i \in I}$ of open subsets of X is an *open cover* of X if $X = \bigcup_{i \in I} U_i$. In other words, every point in X lies in some set U_i .

A sub-collection $\{U_i\}_{i \in I_0}$ (where $I_0 \subseteq I$) is an *open subcover* (or simply *subcover*) if $X = \bigcup_{i \in I_0} U_i$. In other words, every point in X lies in some set U_i in the subcover.

Definition 1.2. (Compact spaces; compact subspaces.) We say that a topological space (X, \mathcal{T}) is *compact* if **every** open cover of X has a finite subcover. A subset $A \subseteq X$ is called *compact* if it is compact with respect to the subspace topology.

Example 1.3. Let (X, \mathcal{T}) be a finite topological space. Then X is compact.

Example 1.4. Let X be a topological space with the indiscrete topology. Then X is compact.

Example 1.5. Let X be an infinite topological space with the discrete topology. Then X is **not** compact.

In-class Exercises

1. Let X be a set with the cofinite topology. Prove that X is compact.
2. Let $X = (0, 1)$ with the topology induced by the Euclidean metric. Show that X is not compact.
3. Let X be a topological space with topology induced by a metric d . Prove that, if X is compact, then X is a bounded metric space.
4. Suppose that (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) are topological spaces, and $f : X \rightarrow Y$ is a continuous map. Show that, if X is compact, then $f(X)$ is a compact subspace of Y . In other words, the continuous image of a compact set is compact.
5. Prove that a closed subset of a compact space is compact.
6. Suppose that (X, \mathcal{T}) is a Hausdorff topological space. Prove that any compact subset A of X is closed.
7. Consider \mathbb{Q} with the Euclidean metric. Show that the subset $(-\pi, \pi) \cap \mathbb{Q}$ of \mathbb{Q} is closed and bounded, but not compact.
8. **Bonus (Optional)**. Let X be a set, and let x_0 be a distinguished element of X . Let

$$\mathcal{T} = \{A \subseteq X \mid x_0 \notin A \text{ or } X \setminus A \text{ is finite}\}.$$

- (a) Show that \mathcal{T} defines a topology on X .
- (b) Verify that (X, \mathcal{T}) is Hausdorff.
- (c) Verify that (X, \mathcal{T}) is compact.

This exercise shows that **any** set X admits a topology making it a compact Hausdorff topological space.

9. **Bonus (Optional)**. Suppose that (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) are topological spaces, and $f : X \rightarrow Y$ is a closed map (this means that $f(C)$ is closed for every closed subset $C \subseteq X$). Suppose that Y is compact, and moreover that $f^{-1}(y)$ is compact for every $y \in Y$. Prove that X is compact.
10. **Bonus (Optional)**. Let $K_1 \supseteq K_2 \supseteq \cdots$ be a descending chain of nonempty, closed, compact sets. Then $\bigcap_{n \in \mathbb{N}} K_n \neq \emptyset$.