1 Products of compact spaces

In this handout, we will prove the following theorem.

Theorem 1.1. (Products of compact spaces). *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Then $X \times Y$ is compact with respect to the product topology $\mathcal{T}_{X\times Y}$ if and only if both X and Y are compact.*

In-class Exercises

1. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces. Suppose that their Cartesian product $X \times Y$ is compact with respect to the product topology $\mathcal{T}_{X\times Y}$. Prove that X and Y are compact.

2. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be compact topological spaces. Let $\mathcal{U}$ be any open cover of $X \times Y$ (with the product topology).
   For this exercise, we will call a subset $A \subseteq X$ *good* if $A \times Y$ is covered by a finite subcollection of open sets in $\mathcal{U}$. Our goal is to show that X is good.

   (a) Suppose that $A_1, \ldots, A_r$ is a finite collection of good subsets of X. Show that their union is good.

   (b) Fix $x \in X$. For each $y \in Y$, explain why it is possible to find open sets $U_y \in X$ and $V_y \in Y$ so that $(x, y) \in U_y \times V_y$ and $U_y \times V_y$ is contained in some open set in $\mathcal{U}$.

   (c) Explain why there is a finite list of points $y_1, \ldots, y_n \in Y$ so that the sets $\{V_{y_1}, \ldots, V_{y_n}\}$ cover Y.

   (d) Define
   $$U_x = U_{y_1} \cap U_{y_2} \cap \cdots \cap U_{y_n}.$$
   Show that U_x is a good set, and is an open subset of X containing x. This shows that every element $x \in X$ is contained in a good open set U_x.

   (e) Consider the collection of open subsets $\{U_x \mid x \in X\}$ of X. Use the fact that X is compact to conclude that X is good.

Bonus Exercises (Optional): Local connectedness and path-connectedness

Definition (Local connectedness). Let $(X, \mathcal{T})$ be a topological space. Then X is *locally connected at a point* $x \in X$ if every neighbourhood U_x of x contains a connected open neighbourhood V_x of x . The space X is *locally connected* if it is locally connected at every point $x \in X$.

Definition (Local path-connectedness). Let $(X, \mathcal{T})$ be a topological space. Then X is *locally connected at a point* $x \in X$ if every neighbourhood U_x of x contains a path-connected open neighbourhood V_x of x . The space X is *locally path-connected* if it is locally path-connected at every point $x \in X$.

1. Let $(X, \mathcal{T})$ be a topological space, and let $x \in X$. Show that if X is locally path-connected at x, then it is locally connected at x. Conclude that locally path-connected spaces are locally connected.

2. Let $X = (0, 1) \cup (2, 3)$ with the Euclidean metric. Show that X is locally path-connected and locally connected, but is not path-connected or connected.

3. Let X be the following subspace of $\mathbb{R}^2$ (with topology induced by the Euclidean metric)

$$X = \bigcup_{n \in \mathbb{N}} \left(\{\tfrac{1}{n}\} \times [0, 1] \right) \quad \bigcup \quad \left(\{0\} \times [0, 1] \right) \quad \bigcup \quad \left([0, 1] \times \{0\} \right).$$

   Show that X is path-connected and connected, but not locally connected or locally path-connected.

4. **(Challenge.)** Consider the natural numbers $\mathbb{N}$ with the cofinite topology. Show that $\mathbb{N}$ is locally connected but not locally path-connected.