1 Homeomorphisms

Definition 1.1. (Homeomorphisms of topological spaces). Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A map $f : X \to Y$ is a *homeomorphism* if

- f is continuous,
- f has an inverse $f^{-1} : Y \to X$, and
- f^{-1} is continuous.

The topological space $(X, \mathcal{T}_X)$ is said to be *homeomorphic* to the topological space $(Y, \mathcal{T}_Y)$ if there exists a homeomorphism $f : X \to Y$.

Two topological spaces are considered "the same" topological space if and only if they are homeomorphic.

In-class Exercises

1. (a) Let $(X, \mathcal{T}_X)$ be a topological space. Show that X is homeomorphic to itself.

   (b) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and $f : X \to Y$ a homeomorphism. Explain why $f^{-1} : Y \to X$ is also a homeomorphism. Conclude that X is homeomorphic to Y if and only if Y is homeomorphic to X. (We simply call the spaces "homeomorphic topological spaces").

   (c) Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$, and $(Z, \mathcal{T}_Z)$ be topological spaces. Show that, if X is homeomorphic to Y, and Y is homeomorphic to Z, then X is homeomorphic to Z.

   This exercise shows that homeomorphism defines an *equivalence relation* on topological spaces.

2. (a) Give an example of topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ and a map $f : X \to Y$ such that f is both continuous and invertible, but such that f^{-1} is not continuous.

   (b) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and $f : X \to Y$ a map. Show that f is a homeomorphism if and only if it is a continuous, inveritble, open map.

3. Determine which of the following properties are preserved by homeomorphism. In other words, suppose $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are homeomorphic topological spaces. For each of the following properties P, prove or give a counterexample to the statement "X has property P if and only if Y has property P".
   (For some properties, you will need to assume that X and Y are metric spaces.)

   (i) discrete topology
   (ii) indiscrete topology
   (iii) Hausdorff
   (iv) regular
   (v) number of connected components
   (vi) path-connected
   (vii) complete
   (viii) sequentially compact
   (ix) compact
   (x) bounded
   (xi) metrizable

   Properties that are preserved by homeomorphisms are called *homeomorphism invariants*, *topological invariants*, or *topological properties* of a topological space.

4. Use the results of Problem 3 to explain why the following pairs of spaces are *not* homeomorphic.
   (a) $(0,1)$ and $[0,1]$ (with the Euclidean metric)
   (b) $\mathbb{R}$ with the Euclidean metric and $\mathbb{R}$ with the cofinite topology
   (c) $(0,2)$ and $(0,1] \cup (2,3)$ (with the Euclidean metric)

5. **(Bonus).** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and $F : X \to Y$ a continuous function. Let G denote the graph of F (as a subspace of $X \times Y$). Prove that G is homeomorphic to X.