

1 Open and closed sets

Definition 1.1. (Open ball of radius r about x_0 .) Let (X, d) be a metric space, and $x_0 \in X$. Let $r \in \mathbb{R}$, $r > 0$. We define the *open ball of radius r about x_0* as the subset of X

$$B_r(x_0) = \{x \in X \mid d(x_0, x) < r\} \subseteq X.$$

Example 1.2. Let $X = \mathbb{R}$ with the usual Euclidean metric $d(x, y) = |x - y|$. What is $B_1(0)$? What is $B_2(-6)$?

Example 1.3. Let $X = \mathbb{R}^2$ with the usual Euclidean metric. Draw $B_2(1, 0)$.

Definition 1.4. (Interior points; open sets in a metric space.)

Let (X, d) be a metric space, and let $U \subseteq X$ be a subset of X . A point $x \in U$ is called an *interior point of U* if there is some radius $r_x \in \mathbb{R}$, $r_x > 0$, so that $B_{r_x}(x) \subseteq U$.

The set $U \subseteq X$ is called *open* if every point $x \in U$ is an interior point of U .

Example 1.5. Show that the interval $[0, 1) \subseteq \mathbb{R}$ is not open.

Example 1.6. Show that the interval $(0, 1) \subseteq \mathbb{R}$ is open.

Proposition 1.7. *Let (X, d) be a metric space, $x_0 \in X$ and $0 < r \in \mathbb{R}$. Then the ball $B_r(x_0)$ is an open subset of X .*

Proof.

Definition 1.8. (Closed sets in a metric space.) A subset $C \subseteq X$ is *closed* if its complement $X \setminus C$ is open.

In-class Exercises

- Let (X, d) be a metric space. Solve (with justification) the following:
 - Is X open?
 - Is \emptyset open?
 - Is X closed?
 - Is \emptyset closed?
- Let $X = \mathbb{R}^2$. Sketch the balls $B_1(0, 0)$ and $B_2(0, 0)$ for each of the following metrics on \mathbb{R}^2 . Denote $\bar{x} = (x_1, x_2)$ and $\bar{y} = (y_1, y_2)$.
 - $d(\bar{x}, \bar{y}) = \|\bar{x} - \bar{y}\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$
 - $d(\bar{x}, \bar{y}) = |x_1 - y_1| + |x_2 - y_2|$
 - $d(\bar{x}, \bar{y}) = \begin{cases} 0, & \bar{x} = \bar{y} \\ 1, & \bar{x} \neq \bar{y} \end{cases}$
- Let $\{U_i\}_{i \in I}$ denote a collection of open sets in a metric space (X, d) .
 - Prove that the union $\bigcup_{i \in I} U_i$ is an open set. Do not assume that I is necessarily finite, or countable!
 - Show by example that the intersection $\bigcap_{i \in I} U_i$ may not be open. (This means, give an example of a metric space (X, d) and a collection of open sets $U_i \subseteq X$, and prove that $\bigcap_{i \in I} U_i$ is not open).
 - Now assume we have a **finite** collection $\{U_i\}_{i=1}^n$ of open sets in a metric space. Prove that the intersection $\bigcap_{i=1}^n U_i$ is open.
- (Optional)**. Consider the metric space \mathbb{R} with the Euclidean metric. Is the subset $\mathbb{Q} \subseteq \mathbb{R}$ open? Is it closed?