1 Continuous functions on metric spaces

Definition 1.1. (Continuous functions $f : \mathbb{R} \to \mathbb{R}$.) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then f is *continuous at a point* $x \in \mathbb{R}$ if ...

The function f is called *continuous* if it is continuous at every point $x \in \mathbb{R}$.

Rephrased:

How can we generalize this definition to general metric spaces?

Definition 1.2. (Continuous functions on metric spaces.) Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f : X \to Y$ be a function. Then f is *continuous at a point* $x \in X$ if ...

The function f is called *continuous* if it is continuous at every point $x \in X$.

Rephrased:

In-class Exercises

1. In this question, we will prove the following result:

   Theorem (Continuous functions.) Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be a function. Then f is continuous if and only if, given any open set $U \subseteq Y$, its preimage $f^{-1}(U) \subseteq X$ is open.

   (a) Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be a continuous function. Suppose that $U \subseteq Y$ is an open set. Prove that $f^{-1}(U)$ is open.

   (b) Suppose that f is a function with the property that, for every open set $U \subseteq Y$, the preimage $f^{-1}(U)$ is an open set in X. Show that f is continuous.

2. Let $f : X \to Y$ and $g : Y \to Z$ be continuous functions between metric spaces. Show that the composite

   $$g \circ f : X \to Z$$

   is continuous. *Hint:* With our new criterion for continuity, this argument can be quite quick!