1 Convergent sequences in metric spaces

Definition 1.1. (Convergent sequences in $\mathbb{R}$.) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Then we say that the sequence *converges* to $a_\infty \in \mathbb{R}$, and write $\lim_{n\to\infty} a_n = a_\infty$, if ...

Definition 1.2. (Convergent sequences in metric spaces.) Let (X, d_X) be a metric space, and let $(a_n)_{n\in\mathbb{N}}$ be a sequence of elements of X. Then we say that the sequence *converges* to $a_\infty \in X$, and write $\lim_{n\to\infty} a_n = a_\infty$, if ...

Rephrased:

In-class Exercises

1. Prove the following result:

   Theorem (An equivalent definition of convergence.) A sequence $(a_n)_{n\in\mathbb{N}}$ of points in a metric space (X, d) converges to a_∞ if and only if for any open set $U \in X$ which contains a_∞, there exists some $N > 0$ so that $a_n \in U$ for all $n \geq N$.

2. Prove the following result:

   Theorem (Another definition of continuous functions.) Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be a function. Then f is continuous if and only if the following condition holds: given any convergent sequence $(a_n)_{n\in\mathbb{N}}$ in X, then $\big(f(a_n)\big)_{n\in\mathbb{N}}$ converges in Y, and

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right).$$

3. **(Optional.)** Let $f : X \to Y$ be a function of sets X and Y. Let $A, B \subseteq X$ and $C, D \subseteq Y$. For each of the following, determine whether you can replace the symbol $\Box$ with $\subseteq, \supseteq, =$, or none of the above.

   (a) $f(A \cap B) \quad \Box \quad f(A) \cap f(B)$

   (b) $f(A \cup B) \quad \Box \quad f(A) \cup f(B)$

   (c) For $A \subseteq B$, $\quad f(B \setminus A) \quad \Box \quad f(B) \setminus f(A)$

   (d) $f^{-1}(C \cup D) \quad \Box \quad f^{-1}(C) \cup f^{-1}(D)$

   (e) $f^{-1}(C \cap D) \quad \Box \quad f^{-1}(C) \cap f^{-1}(D)$

   (f) For $C \subseteq D$, $\quad f^{-1}(D \setminus C) \quad \Box \quad f^{-1}(D) \setminus f^{-1}(C)$

   (g) $A \quad \Box \quad f^{-1}(f(A)))$

   (h) $C \quad \Box \quad f(f^{-1}(C)))$