1 The interior and the closure of a set

Definition 1.1. (Interior of a set.) Let (X, d) be a metric space, and $A \subseteq X$ a subset. Then the *interior of* A, denoted $\mathring{A}$, is defined to be the set

$$\mathring{A} = \{a \in A \mid a \text{ is an interior point of } A\}.$$

Note that $\mathring{A} \subseteq A$. We will see in the exercises that $\mathring{A}$ is an open set, and it is in a sense the largest open subset of A.

Definition 1.2. (Neighbourhood of a point x.) Let (X, d) be a metric space, and $x \in X$. Then any open set U containing x is called a *neighbourhood of* x.

Definition 1.3. (Closure of a set.) Let (X, d) be a metric space, and $A \subseteq X$ a subset. Then the *closure of* A, denoted $\overline{A}$, is defined to be the set

$$\overline{A} = \{x \in X \mid \text{every neighbourhood } U \text{ of } x \text{ contains a point of } A\}.$$

Example 1.4. What is the closure of the open set $B_1(0, 0) \subseteq \mathbb{R}^2$?

Show that $\overline{A}$ consists of two kinds of points:

1. Elements of A,
2. Elements of $X \setminus A$ that are accumulation points of A.

We will see that $\overline{A}$ is a closed set, and that in a sense it is the smallest closed set containing A.

In-class Exercises

1. Prove the following Theorem.

   Theorem 1.5. *Let (X, d) be a metric space, and $A \subseteq X$ a subset.*

   (i) $\mathring{A} \subseteq A$
   (ii) *A is open if and only if* $A = \mathring{A}$
   (iii) *If $A \subseteq B$ then* $\mathring{A} \subseteq \mathring{B}$
   (iv) $\mathring{\mathring{A}} = \mathring{A}$
   (v) *$\mathring{A}$ is open in X*
   (vi) *$\mathring{A}$ is the largest open subset of A in the following sense: If $U \subseteq A$ is any open subset of A, then $U \subseteq \mathring{A}$*

2. Prove the following Theorem.

   Theorem 1.6. *Let (X, d) be a metric space, and $A \subseteq X$ a subset.*

   (i) $A \subseteq \overline{A}$
   (ii) *If $A \subseteq B$ then* $\overline{A} \subseteq \overline{B}$
   (iii) *A is closed if and only if* $A = \overline{A}$
   (iv) $\overline{\overline{A}} = \overline{A}$
   (v) *$\overline{A}$ is closed in X*
   (vi) *$\overline{A}$ is the smallest closed set containing A, in the following sense: If $A \subseteq C$ for some closed set C, then $\overline{A} \subseteq C$*

3. **(Optional).** Let A be a subset of a metric space (X, d). For each of the following statements, either prove the statement, or construct a counterexample.

   (a) $\overset{\circ}{\widehat{X \setminus A}} \subseteq X \setminus \mathring{A}$ (b) $\overset{\circ}{\widehat{X \setminus A}} \supseteq X \setminus \mathring{A}$ (c) $\overline{X \setminus A} \subseteq X \setminus \overline{A}$ (d) $\overline{X \setminus A} \supseteq X \setminus \overline{A}$

4. **(Optional).** Let A_i, $i \in I$, be a collection of subsets of a metric space (X, d). For each of the following statements, either prove the statement, or construct a counterexample.

   (a) $\overset{\circ}{\overbrace{\bigcup_{i \in I} A_i}} \subseteq \bigcup_{i \in I} \mathring{A}_i$ (c) $\overline{\bigcup_{i \in I} A_i} \subseteq \bigcup_{i \in I} \overline{A_i}$ (e) $\overset{\circ}{\overbrace{\bigcap_{i \in I} A_i}} \subseteq \bigcap_{i \in I} \mathring{A}_i$ (g) $\overline{\bigcap_{i \in I} A_i} \subseteq \bigcap_{i \in I} \overline{A_i}$

   (b) $\overset{\circ}{\overbrace{\bigcup_{i \in I} A_i}} \supseteq \bigcup_{i \in I} \mathring{A}_i$ (d) $\overline{\bigcup_{i \in I} A_i} \supseteq \bigcup_{i \in I} \overline{A_i}$ (f) $\overset{\circ}{\overbrace{\bigcap_{i \in I} A_i}} \supseteq \bigcap_{i \in I} \mathring{A}_i$ (h) $\overline{\bigcap_{i \in I} A_i} \supseteq \bigcap_{i \in I} \overline{A_i}$