

1 Topological spaces

Definition 1.1. (Topology; Topological space.) Let X be a set. A *topology* \mathcal{T} on X is a collection of subsets of X that satisfies the following properties:

(T1) The sets \emptyset and X are elements of \mathcal{T} .

(T2) If $\{U_i\}_{i \in I}$ is any collection of elements of \mathcal{T} , then $\bigcup_{i \in I} U_i$ is in \mathcal{T} .

(T3) If $U, V \in \mathcal{T}$, then $U \cap V \in \mathcal{T}$.

A set X endowed with a topology \mathcal{T} is called a *topological space* and denoted by (X, \mathcal{T}) , or simply denoted by X when \mathcal{T} is clear from context. The elements of \mathcal{T} are called the *open subsets* of the topological space X .

We have already proved the following result:

Theorem 1.2. (Metrics induce a topology.) Let (X, d) be a metric space. Then the collection \mathcal{T}_d of all open sets in X forms a topology on X .

The topology \mathcal{T}_d is called the *topology induced by the metric* d . We will see that not every topology on a set X necessarily arises from a metric structure on X .

Definition 1.3. (Metrizable topologies.) A topology \mathcal{T} on a set X is said to be *metrizable* if there exists a metric d on X such that \mathcal{T} is the set \mathcal{T}_d of open sets for the metric space (X, d) .

In-class Exercises

- Let X be a set.
 - Let $\mathcal{T} = \{\emptyset, X\}$. Prove that \mathcal{T} is a topology on X . It is called the *indiscrete topology*.
 - Let \mathcal{T} be the collection of all subsets of X . Prove that \mathcal{T} is a topology on X . It is called the *discrete topology*.
- Let X be a set.
 - Show that the discrete topology is metrizable.
 - Suppose that X contains at least 2 elements. Show that the indiscrete topology is not metrizable.
- Let (X, \mathcal{T}) be a topological space. Show that $V \in \mathcal{T}$ if and only if for every $x \in V$ there is some set $U_x \subseteq X$ containing x such that $U_x \in \mathcal{T}$ and $U_x \subseteq V$.
- Consider the following definition.

Definition 1.4. (Closed subsets of a topological space.) Let (X, \mathcal{T}) be a topological space. A subset $C \subseteq X$ is called *closed* if its complement is open, that is, if $X \setminus C$ is an element of \mathcal{T} .

- Verify that X and \emptyset are closed.
- Suppose that B and C are closed subsets of X . Verify that $B \cup C$ is closed.
- Suppose that $\{C_i\}_{i \in I}$ is a collection of closed sets in X . Verify that $\bigcap_{i \in I} C_i$ is closed.